On the classification of nuclear C^*-algebras

Marius Dadarlat* Søren Eilers†

Final version, August 1998

Contents

*Partially supported by the National Science Foundation, # 9622434-DMS
†Partially supported by the Carlsberg Foundation

1 Introduction

Two of the most influential works on C^*-algebras from the mid-seventies — Brown, Douglas and Fillmore's [BDF77] and Elliott's [Ell76] — both contain *uniqueness* and *existence* results in the now standard sense which we shall outline below. These papers served as keystones for two separate theories —*KK-theory* and *the classification program* — which for many years parted ways with only moderate interaction. But with this common origin in mind, it is not surprising that recent years have seen a fruitful interaction which has been one of the main engines behind rapid progress in the classification program.

In the present paper we take this interaction even further. We prove general existence and uniqueness results using KK-theory and a concept of quasidiagonality for representations. These results are employed to obtain new classification results for certain classes of quasidiagonal C^*-algebras introduced by H. Lin. An important novel feature of these classes is that they are defined by a certain local approximation property, rather than by an inductive limit construction.

Our existence and uniqueness results are in the spirit of the classical Ext-theory from [BDF77]. The main complication overcome in the paper is to control the stabilization which is necessary when one works with finite C^*-algebras. In the infinite case, where programs of this type have already been successfully carried out, stabilization is unnecessary. Yet, our methods are sufficiently versatile to allow us to reprove, from a handful of basic results, the classification of purely infinite nuclear C^*-algebras of Kirchberg and Phillips.

Indeed, it is our hope that this can be the starting point of a unified approach to classification of nuclear C^*-algebras.

Apart from KK-theory, the main technical tools are approximate morphisms and partial KK-elements, defined using K-theory with coefficients.

1.1 Existence and uniqueness

Existence and uniqueness theorems can be found in most, if not all, classification papers to combine with refinements of Elliott's intertwining argument to yield classification theorems.

In this framework, a **uniqueness result** allows one to conclude that if the K-theoretical elements induced by two $*$-homomorphisms $\varphi, \psi : A \longrightarrow B$ coincide, with A, B C^*-algebras satisfying extra conditions, then φ and ψ are equivalent in a sense which generalizes unitary equivalence. In [Ell76], one concludes from $\varphi_* = \psi_*$ on $K_0(A)$ that φ and ψ are approximately unitarily equivalent when A and B are both AF (cf. [Bra72]). In [BDF77] one shows that if $[\varphi]_{KK} = [\psi]_{KK}$, then φ and ψ are unitarily equivalent whenever $A = C(X)$ and $B = \mathcal{L}(H)/\mathcal{K}(H)$.

The role of the equally important **existence results** is to provide means of realizing given K-theoretical data by a sufficiently multiplicative completely positive contractive map. In [Ell76], one realizes any positive element of $\mathrm{Hom}(K_0(A), K_0(B))$ by a $*$-homomorphisms when A and B are AF. In [BDF77], any element of $KK(C(X), \mathcal{L}(H)/\mathcal{K}(H))$ is realized by a $*$-monomorphism.

One of the main obstacles in achieving general existence and uniqueness results has been the fact that, as soon as one ventures beyond these classical examples, one only can expect

to achieve *stable* versions of such theorems in a sense that involves adding or subtracting maps of the form $\mu : A \longrightarrow \mathbf{M}_n(B)$. The challenge has been to control the complexity of the stabilization, in order to be able to incorporate them into classification results.

This has been quite successfully carried out, using maps μ with finite-dimensional images, in important classes of quasidiagonal C^*-algebras. But in most existence and uniqueness results so far the source A has been required to be a member of a small class of C^*-algebras forming *building blocks* for the class of C^*-algebras one has tried to classify, leading to restrictions on the ensuing classification results.

We offer, in the present paper, existence and uniqueness results valid for sources way beyond even the full class of nuclear quasidiagonal C^*-algebras. More precisely, we consider a unital and nuclear source A and a unital target B, and require that this pair allows an *absorbing* and *quasidiagonal* unital representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ as defined in Section 3.1 below. Such a map exists automatically when A is quasidiagonal, or (by a theorem of Lin [Lin97]) when A can be embedded into B via a third simple C^*-algebra.

In this case, our uniqueness result Theorem 3.4.1 states that if $\varphi, \psi : A \longrightarrow B$ are two $*$-homomorphisms inducing the same KK-class, we may conclude that φ is stably approximately unitarily equivalent to ψ in a sense involving adding "finite pieces" of γ. For C^*-algebras A which satisfy the universal coefficient theorem of [RS87], our result is predated by one of a similar nature, valid when A or B is simple, which appears in [Lin97]. We emphasize that our uniqueness result does not depend on the universal coefficient theorem, and we believe that in addition to proving a more general result, our arguments are somewhat more conceptual.

In fact, we shall require uniqueness results which hold also for sufficiently multiplicative completely positive contractive maps, and this entails the problem of associating to such maps a kind of *partial* KK-elements to substitute for the globally defined group homomorphisms one gets from $*$-homomorphisms. This is done using the universal multicoefficient theorem of [DL96b] in a fashion explained by the first named author in his talk at the *Workshop on the classification of amenable C^*-algebras* at the Fields Institute in December of 1994. As soon as we have made sense out of this concept, a uniqueness result for completely positive contractive maps, Theorem 4.1.4, can be derived from those for $*$-homomorphisms using a procedure originating with [LP95] in the torsion free case. This method also requires keeping a close eye on the K-theory for products of C^*-algebras.

In our existence result Theorem 5.1.5 we manage to realize – partially, in a sense corresponding to the one used in the uniqueness result – a given element from $KK(A, B)$ as a difference of completely positive contractive maps from A to $\mathbf{M}_N(B)$. Again, all we require is that A is nuclear, and that an absorbing quasidiagonal representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ exists as in the uniqueness case. Furthermore, one of the maps in the difference can be chosen as a "finite piece" of γ.

In the building block approach, establishing existence has typically been somewhat easier than achieving uniqueness results. This may still be the case in our setting, but at the current stage it is in fact the existence which is causing problems. Indeed, the existence result offered in our paper has shortcomings in the finite case which are responsible for a number of unwanted, and hopefully redundant, restrictions in the resulting classification results. For instance, we do not have sufficient technology to prove in full generality that a

positive KK-element can be realized by a single map, as one would expect to be the case.

As a main application, we apply our existence and uniqueness results to the class of TAF C^*-algebras introduced and studied by H. Lin. Using a factorization property of these C^*-algebras, we are able to prove in Theorem 6.2.4 that, up to an isomorphism, there is only one unital, separable, nuclear and simple TAF C^*-algebra satisfying the UCT with $K_0(A) = \mathbb{Q}$ and $K_1(A) = G$, where G is a countable fixed arbitrary group, thus proving that every such C^*-algebra falls in the well studied class of AD algebras of real rank zero.

We believe, however, that our existence and uniqueness results will be applicable to a wide range of classification problems, extending well beyond the TAF case. In fact, it might be that at least the uniqueness result will be sufficiently versatile to serve as a unifying element for many future efforts to classify nuclear C^*-algebras. To substantiate this claim, we apply our methods to the case of purely infinite C^*-algebras, reproving in Theorem 6.3.6, rather easily, the classification theorem of purely infinite simple unital nuclear C^*-algebras (see [Kir94] and [Phi94]) from a handful of fundamental results about such algebras.

We reported on the present paper at the 1998 GPOTS. At the same conference, H. Lin reported results which – although the methods differ – overlap with our classification results in the TAF C^*-algebra case. More details are given in the notes of the present paper.

1.2 Methods

To arrive at uniqueness and existence results from the existence of an absorbing and quasidiagonal representation, we depend on the full force of KK-theory, including several of the different realizations of the Kasparov groups and their interrelations.

To achieve such representations, one may use the Kasparov Weyl-von Neumann-Voiculescu theorem along with the concept of quasidiagonality. One can also employ a related result by H. Lin [Lin97] stating that a certain extension associated to a unital inclusion $\iota : A \longrightarrow B$ is absorbing when A is nuclear and B is simple.

To refine uniqueness results we depend on a number of basic properties about the K-theory of products of C^*-algebras. Furthermore, to apply our results to classify TAF C^*-algebras, we use several results about structural properties of such C^*-algebras by Lin. We prove classification for purely infinite algebras by appealing to the embedding theorem for exact C^*-algebras of [KP98], as well as structural results by Kirchberg, Phillips and Rørdam.

1.3 Organization

The paper is organized as follows. In Section 2 we lay out notation and define several relevant classes of C^*-algebras. Then in Section 3, we establish (without using the universal coefficient theorem) the uniqueness result which is at the core of the paper. This is subsequently refined (using, among other things, the universal coefficient theorem) in Section 4. The basic existence results are collected in Section 5. The applications to classification are presented in Sections 6.2 and 6.3, concerning the TAF and the purely infinite C^*-algebras, respectively. Finally, Appendix A contains results about the K-theory of products of C^*-algebras and an explanation of how to associate partial KK-elements to sufficiently multiplicative completely positive maps.

2 General preliminaries

2.1 Notation

Some classes of C^*-algebras

We single out several classes of C^*-algebras for easy reference:

Definition 2.1.1 We say that a separable C^*-algebra *satisfies the UCT* if the diagram

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \longrightarrow KK(A,B) \longrightarrow \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

is a short exact sequence for every σ-unital algebra B.

A large class of algebras satisfying UCT was exhibited in [RS87]. It is not known whether there exist separable nuclear C^*-algebras not satisfying the UCT. If the separable C^*-algebra A satisfies the UCT, then for any σ-unital C^*-algebra B the sequence:

$$(2.1) \qquad 0 \longrightarrow \mathrm{Pext}(K_*(A), K_{*+1}(B)) \longrightarrow KK(A,B) \longrightarrow \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B)) \longrightarrow 0$$

is also exact by [DL96a]. Here $\underline{\mathbf{K}}(-)$ denotes the sum of all K-theory groups with $\mathbb{Z}/n$ coefficients, $n \geq 1$, and Λ denotes the natural set of coefficient transformations and the Bockstein operations (see [Sch84] and [DL96a]).

Definition 2.1.2 We say that a unital $*$-homomorphism $\iota : A \longrightarrow B$ is a *unital simple embedding* if it can be factored

$$A \overset{\iota'}{\hookrightarrow} C \overset{\iota''}{\hookrightarrow} B$$

where ι', ι'' are injective and C is simple.

This implies that C, ι' and ι'' are unital. Note that the composition (on either side) of a simple embedding with a unital injective $*$-homomorphism is a simple embedding.

If B is a C*-algebra we denote by $\mathrm{Proj}(B)$ the set of all selfadjoint projections in B. The K-theory class of a projection p is denoted by $[p] \in K_0(B)$. If B is unital we let $\mathcal{U}_n(B)$ denote the unitary group of $\mathbf{M}_n(B)$.

Definition 2.1.3 A C^*-algebra B is called an *admissible target algebra* if it is unital, has real rank zero ([BP91]) and satisfies

(i) whenever $p, q \in \mathrm{Proj}(B \otimes \mathcal{K})$, then $[p] = [q] \Longrightarrow p \oplus 1_B \sim q \oplus 1_B$
(ii) the canonical map $\mathcal{U}_1(B) \longrightarrow K_1(B)$ is surjective

and if either

(iii.∞) the canonical map $\mathrm{Proj}(B) \longrightarrow K_0(B)$ is surjective

or both of

(iii.1) For any $x \in K_0(B)$ such that $nx \geq 0$ for some $n \geq 0$, one has $x + [1_B] \geq 0$.

(iv.1) For any $x \in K_0(B)$ and any $n \neq 0$, there is $y \in K_0(B)$ such that $-[1_B] \leq y \leq [1_B]$ and $x - y \in nK_0(B)$.

holds.

When needed, we distinguish between admissible target algebras satisfying (iii.1)–(iv.1) or (iii.∞) by calling them admissible of *finite type* or *infinite type*, respectively. Examples will be given in Propositions 6.1.6 and 6.3.2.

The point of this definition is that whenever a sequence of admissible targets are given, then both of the natural maps

$$\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(\prod B_i)) \longrightarrow \prod \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B_i)) \tag{2.2}$$

$$KK(A, \prod B_i / \sum B_i) \longrightarrow \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(\prod B_i / \sum B_i)) \tag{2.3}$$

will be injective (the latter in fact an isomorphism) for an A satisfying the UCT. We defer the proof of this to Appendix A.1 below. We are also going to need that whenever B_i is a sequence of admissible target algebras, then $\prod B_i / \sum B_i$ is admissible.

Remark 2.1.4 In fact, we can get injectivity for the maps discussed above asserting considerably weaker versions of (iii)-(iv). This will be clear from Appendix A.1. On the other hand, $B_i = C([0,1])$ is a counterexample to injectivity in (2.2) and $B_i = \mathbb{C}$ a counterexample to (2.3).

2.2 KK-theory

We depend on Kasparov's KK-theory from [Kas80b] for most of this paper, as well on several different characterizations or realizations of it. As a standard picture of a KK-cycle, we shall adopt the one from [Hig87a, 2.1]. There $KK(A, B)$ is defined in terms of triples $(\varphi_+, \varphi_-, x)$ where $\varphi_\pm : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ are $*$-homomorphisms and $x \in M(\mathcal{K}(H) \otimes B)$ satisfies

$$x\varphi_+(a) - \varphi_-(a)x \in \mathcal{K}(H) \otimes B \tag{2.4}$$

$$\varphi_+(a)(x^*x - 1), \varphi_-(a)(xx^* - 1) \in \mathcal{K}(H) \otimes B \tag{2.5}$$

for each $a \in A$. Higson works with separable C^*-algebras only, but his picture extends readily to the case of a σ-unital B. Equally important to us is the related Cuntz picture of KK-theory, studied in [Hig87b], which we shall indicate by KK_h as in Chapter 4 of [JT91]. Here the cycles are *Cuntz pairs* (φ_+, φ_-) satisfying

$$\varphi_+(a) - \psi_-(a) \in \mathcal{K}(H) \otimes B.$$

3 Uniqueness up to absorption

In the first section of the paper we establish, via Ext- and KK-theory, a uniqueness result proving that two KK-equivalent completely positive contractive maps are approximately unitarily equivalent after one adds an absorbing representation. Most of the work goes into controlling the nature of the unitary implementing this equivalence, proving that it interacts sufficiently well with the other components to allow truncation in a sense to be made precise in Section 3.4.

3.1 Preliminaries

Notation and conventions

In all of Section 3, we only work with infinite, separable Hilbert spaces, so all Hilbert spaces in this paper are isomorphic. However, we introduce the following notation to aid the reader in distinguishing between different instances of them. We start with a separable Hilbert space H_1 and define

$$H_m = \overbrace{H_1 \oplus \cdots \oplus H_1}^{m}.$$

for any $m \in \mathbb{N}$. There are now canonical identifications between, say, $\mathbf{M}_2(\mathcal{K}(H_1) \otimes B)$ and $\mathcal{K}(H_2) \otimes B$, and we shall employ them tacitly in the following. However, we choose *not* to apply the (non-canonical) isomorphisms between, e.g., $\mathcal{K}(H_1) \otimes B$ and $\mathcal{K}(H_2) \otimes B$, as we feel this helps to clarify our constructions. We shall abandon this practice towards the end of this section.

We work with the multiplier algebras $M(\mathcal{K}(H_m) \otimes B)$, as well as the corona algebras

$$Q(\mathcal{K}(H_m) \otimes B) = M(\mathcal{K}(H_m) \otimes B)/\mathcal{K}(H_m) \otimes B.$$

The quotient map from $M(\mathcal{K}(H_m) \otimes B)$ to $Q(\mathcal{K}(H_m) \otimes B)$ is denoted by π_m, and whenever there is need for distinction, we write 1_m and 0_m for the identity and zero elements of these algebras.

Definition 3.1.1 An *admissible scalar representation* $\theta : A \longrightarrow M(\mathcal{K}(H_1) \otimes B)$ is a $*$-homomorphism which factors as

$$A \xrightarrow{\theta'} \mathcal{L}(H) \xrightarrow{-\otimes \mathrm{id}} \mathcal{L}(H) \otimes M(B) \hookrightarrow M(\mathcal{K}(H) \otimes B)$$

where θ' is unital, faithful and of infinite multiplicity, i.e. of the form $\infty \cdot \gamma$ for some representation γ.

When θ is a admissible scalar representation, we are also going to consider representations of the form

$$0_{m-1} \oplus \theta : A \longrightarrow M(\mathcal{K}(H_m) \otimes B)$$

for $m \geq 2$. When the size of the added zero is clear from the context, or irrelevant, we denote this representation by Θ. Note that by convention, Θ is *never* unital.

Absorbing and quasidiagonal representations

Definition 3.1.2 Fix a unital C^*-algebra B. When $\gamma : A \longrightarrow \mathcal{L}_B(E)$ and $\gamma' : A \longrightarrow \mathcal{L}_B(E')$ are two unital representations, with E and E' Hilbert C^*-modules over B, we say that γ and γ' are *equivalent*, and write $\gamma \sim \gamma'$, if there exists a sequence $U_m \in \mathcal{L}_B(E, E')$, consisting of unitaries, such that

(i) $\|\gamma(a) - U_m\gamma'(a)U_m^*\| \longrightarrow 0$, $m \longrightarrow \infty$
(ii) $\gamma(a) - U_m\gamma'(a)U_m^* \in \mathcal{K}_B(E)$

for any $a \in A$.

Definition 3.1.3 A unital representation $\gamma : A \longrightarrow \mathcal{L}_B(E)$ is *absorbing* if for any other unital representation $\gamma' : A \longrightarrow \mathcal{L}_B(E')$, $\gamma \oplus \gamma' \sim \gamma$.

Clearly any two absorbing representations are equivalent. Kasparov proved in [Kas80a], generalizing Voiculescu's result in [Voi76], that any admissible scalar representation is absorbing if A is separable and nuclear. Another class of absorbing representations was exhibited by Lin in [Lin97], based on unital inclusions of A into B, where either A or B is simple. The observation that Lin's proof carries over to the case of unital simple embeddings (cf. 2.1.2) is so crucial to our approach that we shall state is as a separate lemma:

Lemma 3.1.4 *When $\iota : A \longrightarrow B$ is a unital simple embedding and A is nuclear, the map $d_\iota : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ given by $d_\iota(a) = 1 \otimes \iota(a)$ is absorbing.*

Proof: By assumption, a simple C^*-algebra C and unital inclusions $\iota' : A \longrightarrow C$ and $\iota'' : C \longrightarrow B$ can be found with $\iota = \iota''\iota'$. By [Lin97, 1.6], $d_{\iota'}$ is absorbing. Since $\mathrm{id} \otimes \iota''$ preserves approximate units, it induces a map $\widehat{\iota''} : M(\mathcal{K} \otimes C) \longrightarrow M(\mathcal{K} \otimes B)$, and by [Lin97, 1.11] (valid since A is nuclear), we conclude that $\widehat{\iota''} d_{\iota'} = d_{\iota''\iota'} = d_\iota$ is also absorbing. ∎

Definition 3.1.5 A representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ is *quasidiagonal* if there exists an approximate unit of projections (e_n) for $\mathcal{K}(H) \otimes B$ with the property that

$$[e_n, \gamma(a)] \longrightarrow 0$$

for all $a \in A$.

If γ is quasidiagonal and $\gamma \sim \gamma'$, then γ' is quasidiagonal. It is clear that if γ is quasidiagonal, one may assume that (e_n) has the property that $e_n \in \mathbf{M}_{r_n}(B)$ for some sequence of integers (r_n).

When this is the case, we may define a sequence of completely positive morphisms $\gamma_n : A \longrightarrow \mathbf{M}_{r_n}(B)$ by $\gamma_n(a) = e_n\gamma(a)e_n$. We call $(\gamma_n)_{n\in\mathbb{N}}$ a *quasidiagonalization* of γ by $(e_n)_{n\in\mathbb{N}}$. Note that the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative.

The concept of relative quasidiagonality that we use in this paper was studied in [Sal92] and [Sch84].

Remark 3.1.6 If there exists an absorbing and quasidiagonal representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$, then *any* absorbing representation will be quasidiagonal.

Note that d_ι of Lemma 3.1.4 is always quasidiagonal, as it commutes with projections $e_n = n \cdot 1_B$. Thus all absorbing representations $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ are quasidiagonal when there is a unital simple embedding of A into B and A is nuclear. The same holds true whenever A is a nuclear quasidiagonal C^*-algebra.

3.2 Consequences of Ext-theory

Obtaining elements of Ext

We work with $*$-homomorphisms $\varphi : A \to \mathcal{K}(H_1) \otimes B$. If $\varphi : A \to B$ is a $*$-homomorphism, then we regard φ as a map into $\mathcal{K}(H_1) \otimes B$ by embedding B as a $(1,1)$-corner of $\mathcal{K}(H_1) \otimes B$. The pullback of the essential, semisplit extension

$$0 \to \mathcal{K}(H_1) \otimes SB \to \mathcal{K}(H_1) \otimes CB \to \mathcal{K}(H_1) \otimes B \to 0 \tag{3.1}$$

by the $*$-homomorphism φ is the mapping cone extension

$$0 \to \mathcal{K}(H_1) \otimes SB \to C_\varphi \to A \to 0 \tag{3.2}$$

This is an essential, semisplit extension. With

$$\chi^\varphi(a)(t) = t\varphi(a), \quad a \in A, \ t \in [0,1],$$

we get a completely positive contractive map $\chi^\varphi : A \to C_b([0,1), \mathcal{K}(H_1) \otimes B)$. Since

$$C_b([0,1), \mathcal{K}(H_1) \otimes B) \subseteq C_{b,\text{strict}}((0,1), M(\mathcal{K}(H_1) \otimes B)) = M(\mathcal{K}(H_1) \otimes SB),$$

we may and shall consider χ^φ as a map into $M(\mathcal{K}(H_1) \otimes SB)$. With χ^φ as above, one checks that $\pi_1 \circ \chi^\varphi : A \longrightarrow Q(\mathcal{K}(H_1) \otimes SB)$ is the Busby invariant of (3.2). Since χ^φ is completely positive and contractive, we have that (3.2) is semisplit as claimed, and hence it defines an element $[C_\varphi] \in \mathrm{Ext}(A, SB)^{-1}$.

Proposition 3.2.1 *Let A, B be C^*-algebras with A separable and B σ-unital. Let $\varphi, \psi : A \to \mathcal{K}(H_1) \otimes B$ be two $*$-homomorphisms. If $[\varphi]_{KK} = [\psi]_{KK}$ in $KK(A, B)$, then $[C_\varphi] = [C_\psi]$ in $\mathrm{Ext}(A, SB)^{-1}$.*

Proof: Since the isomorphism $\gamma : \mathrm{Ext}(X,Y)^{-1} \to KK^1(X,Y)$ of Kasparov is natural, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(B,SB)^{-1} & \xrightarrow{\varphi^*} & \mathrm{Ext}(A,SB)^{-1} \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\gamma} \\ KK^1(B,SB) & \xrightarrow{\varphi^*} & KK^1(A,SB) \end{array}$$

and a similar diagram for ψ. If x is the class of (3.1) in $\mathrm{Ext}(B, SB)^{-1}$, then $[C_\varphi] = \varphi^*(x)$ and $[C_\psi] = \psi^*(x)$. Since γ is an isomorphism, in order to show that $[C_\varphi] = [C_\psi]$ it suffices

to prove $\gamma\varphi^*(x) = \gamma\psi^*(x)$. Using the commutative diagram above, this is equivalent to showing that $\varphi^*\gamma(x) = \psi^*\gamma(x)$. By [Bla86, 18.7.2], for any $y \in KK^1(B, SB)$, $\varphi^*(y)$ equals the Kasparov product $[\varphi]_{KK} \otimes y$. Therefore

$$\varphi^*\gamma(x) = [\varphi]_{KK} \otimes \gamma(x) = [\psi]_{KK} \otimes \gamma(x) = \psi^*\gamma(x).$$

■

Weak uniqueness

The starting point of our investigation of uniqueness of maps between two C^*-algebras A and B will be two $*$-homomorphisms, φ and ψ. We will require A to be separable, but it is crucial for applications in Section 4 that B can be any σ-unital C^*-algebra.

Proposition 3.2.2 *Let A be a unital, separable, nuclear C^*-algebra and let B be a σ-unital C^*-algebra. Assume that φ and ψ are two $*$-homomorphisms from A to $\mathcal{K}(H_1) \otimes B$ which satisfy $[\varphi]_{KK} = [\psi]_{KK}$ in $KK(A, B)$. Then for any admissible scalar representation $\theta : A \to M(\mathcal{K}(H_1) \otimes B)$, there exists a strictly continuous map*

$$u : (0,1) \longrightarrow \mathcal{U}(M(\mathcal{K}(H_3) \otimes B))$$

with the property that

$$u_t \begin{bmatrix} t\varphi_t(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} {u_t}^* - \begin{bmatrix} t\psi_t(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} \in C_0((0,1), \mathcal{K}(H_3) \otimes B)$$

for all $a \in A$.

Proof: By Proposition 3.2.1 we conclude that $\pi_1 \circ \chi^\varphi$ and $\pi_1 \circ \chi^\psi$ define the same element of $\operatorname{Ext}(A, \mathcal{K}(H_1) \otimes SB)$. Since $0_1 \oplus \theta$ defines an absorbing extension by Kasparov's Voiculescu theorem [Kas80a], we have that $\pi_3 \circ (\chi^\varphi \oplus 0_1 \oplus \theta)$ is equivalent to $\pi_3 \circ (\chi^\psi \oplus 0_1 \oplus \theta)$. This means that

$$\operatorname{Ad}_{\pi_3(u)} \circ \pi_3 \circ (\chi^\varphi \oplus 0_1 \oplus \theta) = \pi_3 \circ \left(\chi^\psi \oplus 0_1 \oplus \theta\right)$$

for some $u \in \mathcal{U}(M(\mathcal{K}(H_3) \otimes SB))$. In $M(\mathcal{K}(H_3) \otimes SB)$, this amounts exactly to saying that the difference considered in the Proposition is an element of $\mathcal{K}(H_3) \otimes SB$. ■

A trivial KK-cycle

Working with u_t and θ as given in Proposition 3.2.2, we are going to consider

$$\chi_0(a) = \Theta(a) = \begin{bmatrix} 0_2 & \\ & \theta(a) \end{bmatrix} \qquad \chi_t(a) = u_t \begin{bmatrix} 0_2 & \\ & \theta(a) \end{bmatrix} {u_t}^*,$$

where $t \in (0,1)$. This way we get a family of $*$-homomorphisms

$$\chi_t : A \to M(\mathcal{K}(H_3) \otimes B), \quad t \in [0,1)$$

whose properties are collected in the following Lemma.

Lemma 3.2.3 *Let u_t and θ be as in Proposition 3.2.2, and fix $t \in (0,1)$. For any $a \in A$, we have*

(i) $\forall s \in [0,t] : \chi_0(a) - \chi_s(a) \in \mathcal{K}(H_3) \otimes B$
(ii) $\chi_0(a) - \chi_s(a)$*, considered as a function from* $[0,t]$ *to* $\mathcal{K}(H_3) \otimes B$*, is norm continuous.*
(iii) $\chi_s(a)$*, considered as a function from* $[0,t]$ *to* $M(\mathcal{K}(H_3) \otimes B)$*, is strictly continuous.*

Proof: We have, with φ and ψ as in Proposition 3.2.2,

$$\begin{aligned}
\chi_0(a) - \chi_s(a) &= \begin{bmatrix} s\psi(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} - \begin{bmatrix} s\psi(a) & \\ & 0_2 \end{bmatrix} - \\
&\quad u_s \begin{bmatrix} s\varphi(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} u_s^* + u_s \begin{bmatrix} s\varphi(a) & \\ & 0_2 \end{bmatrix} u_s^* \\
&= u_s \begin{bmatrix} s\varphi(a) & \\ & 0_2 \end{bmatrix} u_s^* - \begin{bmatrix} s\psi(a) & \\ & 0_2 \end{bmatrix} + R_s(a),
\end{aligned}$$

where $R_s(a) \in C_0((0,1), \mathcal{K}(H_3) \otimes B)$ by Proposition 3.2.2. Since the first two terms in the last expression lie in $C_0((0,t], \mathcal{K}(H_3) \otimes B)$, (i) and (ii) follow. That (iii) holds follows from the fact that $s \mapsto u_s$ is strictly continuous. ■

Proposition 3.2.4 *Let u_t and θ be as in Proposition 3.2.2, with $\Theta = 0_2 \oplus \theta$. For any fixed $t \in (0,1)$, $[\Theta, \Theta, u_t]$ defines a trivial element of $KK(A,B)$.*

Proof: Fix $t \in (0,1)$. We first note, comparing Lemma 3.2.3(i) to [JT91, 4.1.1], that (χ_0, χ_t) defines a cycle in $KK_h(A,B)$ (cf. Section 2.2). In fact, we get from (ii) and (iii) of Lemma 3.2.3 that $(\chi_0, \chi_s)_{0 \le s \le t}$ forms a homotopy, in the sense of [JT91, 4.1.2], from (χ_0, χ_0) to (χ_0, χ_t). Since $[\chi_0, \chi_0] = 0$, by [JT91, 4.1.4] we conclude that $[\chi_0, \chi_t] = 0$ in $KK_h(A,B)$. Applying the isomorphism $\mu_h : KK_h(A,B) \to KK(A,B)$ considered in [JT91, 4.1.8], we get that $[\chi_t, \chi_0, 1]$ is trivial in $KK(A,B)$. Finally we note that, as explained for instance in [Hig90, 2.3],

$$[\chi_t, \chi_0, 1] = [\mathrm{Ad}_{u_t} \circ \chi_0, \chi_0, 1] = [\chi_0, \chi_0, u_t] = [\Theta, \Theta, u_t].$$

■

3.3 Uniqueness up to absorption

The preceding section left us with a trivial KK-element $[\Theta, \Theta, u]$ where u was a unitary. Using results by Skandalis we now conclude that in a very specific and rather subtle sense, the K_1-class of u is also trivial. The triviality translates to the fact that u induces an approximately inner automorphism on an auxiliary C^*-algebra defined by Lin.

A trivial K_1-element

In this section we work with $\Theta = 0_2 \oplus \theta : A \to \mathcal{L}_B(H_3 \otimes B) = M(\mathcal{K}(H_3) \otimes B)$, where θ is an admissible scalar representation. Recall that A is unital and $\theta(1) = 1$. We use the dot as a shorthand to indicate composition by the quotient maps $\pi : M(\mathcal{K}(H) \otimes B) \longrightarrow$

$Q(\mathcal{K}(H)\otimes B)$. Thus, $\dot\Theta = \pi_3 \circ \Theta$ maps from A to $Q(\mathcal{K}(H_3)\otimes B)$. Furthermore, if $X \subseteq Q(\mathcal{K}(H)\otimes B)$, we denote by X^c the commutator of X in $Q(\mathcal{K}(H)\otimes B)$.

We define a C^*-algebra D_Θ by

$$D_\Theta = \{b \in M(\mathcal{K}(H_3)\otimes B) \mid [b,\Theta(A)] \subset \mathcal{K}(H_3)\otimes B\}.$$

One checks directly that

$$0 \longrightarrow \mathcal{K}(H_3)\otimes B \xrightarrow{j} D_\Theta \xrightarrow{\pi_3} \dot\Theta(A)^c \longrightarrow 0 \tag{3.3}$$

is a short exact sequence of C^*-algebras.

Proposition 3.3.1 *With A and Θ as above, assume that $[\Theta,\Theta,u]=0$ in $KK(A,B)$, where $u\in M(\mathcal{K}(H_3)\otimes B)$ is a unitary. Then $u \in D_\Theta$, and*

$$[u]_{K_1(D_\Theta)} \in j_*(K_1(\mathcal{K}(H_3)\otimes B)),$$

where j is the inclusion in (3.3).

Proof: Being part of a KK-cycle, u must commute with Θ modulo the compacts, and hence $u\in D_\Theta$. We note that we have set up our KK-cycle to be covered by the description of $KK(A,B)$ given in Proposition 2.6 of [Ska88], where now A is considered as a trivially graded C^*-algebra. Indeed, since our Θ can substitute as Skandalis' $\pi\otimes 1$, we may conclude from $[\Theta,\Theta,u]=[\Theta,\Theta,1_3]$ that, by Proposition 2.6 of [Ska88],

$$[\Theta\oplus\Theta,\Theta\oplus\Theta,u\oplus 1_3] \sim_{oh} [\Theta\oplus\Theta,\Theta\oplus\Theta,1_3\oplus 1_3],$$

where $\sim_{oh}$ denotes operator homotopy. This means that there is a norm continuous path of operators $\omega_s \in M(\mathcal{K}(H_6)\otimes B)$, $s\in[0,1]$ with

$$u\oplus 1_3 \overset{\omega_s}{\rightsquigarrow} 1_3 \oplus 1_3\ .$$

which satisfies, with $\Theta_2 = \Theta\oplus\Theta$

$$[\Theta_2(a),\omega_s] \in \mathcal{K}(H_6)\otimes B \tag{3.4}$$

$$\Theta_2(a)(\omega_s\omega_s^*-1), \Theta_2(a)(\omega_s^*\omega_s-1) \in \mathcal{K}(H_6)\otimes B \tag{3.5}$$

for all $s\in[0,1]$. We set $z=u\oplus 1_3$, $e=\dot\Theta_2(1)$ and $e^\perp = 1_6 - e$, and abbreviate

$$C = \dot\Theta_2(A)^c e,$$

noting that this is a C^*-algebra since e and $\dot\Theta_2(A)^c$ commute. Note that $\dot\Theta_2(A)^c = C + e^\perp Q(\mathcal{K}(H_6)\otimes B)e^\perp \cong C + Q(\mathcal{K}(H_4)\otimes B)$, and define $w = e\dot z e$ and $w^\perp = e^\perp \dot z e^\perp$. We have that $[e,\dot\omega_s]=0$ by (3.4), so if we let

$$v_s = e\dot\omega_s e \qquad v_s^\perp = e^\perp\dot\omega_s e^\perp,$$

we will get that $\dot{\omega}_s = v_s \oplus v_s^\perp$ for all s. Furthermore, v_s is a continuous path of unitaries by (3.5), going from w to e in C. This implies that $[w]_C = 0$, and consequently that $[w]_Q = 0$. Since $[u]_{M(\mathcal{K}(H_3)\otimes B)} = 0$, we get

$$0 = [\dot{z}]_Q = [w]_Q + \left[w^\perp\right]_Q = \left[w^\perp\right]_Q .$$

We can now choose n such that there exist a homotopy

$$w^\perp \oplus ne^\perp = v_0^\perp \oplus ne^\perp \overset{w_s}{\rightsquigarrow} (n+1)e^\perp$$

in $\mathcal{U}_{n+1}(e^\perp Q(\mathcal{K}(H_6) \otimes B)e^\perp)$. Adding up $(v_s,\, \omega_s,\, ne)$, and identifying appropriately, we get a homotopy

$$\dot{z} \oplus n1_{Q(\mathcal{K}(H_6)\otimes B)} \overset{\Omega_s}{\rightsquigarrow} (n+1)1_{Q(\mathcal{K}(H_6)\otimes B)}$$

in $\mathcal{U}_{n+1}(C + e^\perp Q(\mathcal{K}(H_6) \otimes B)e^\perp)$. But since $C + e^\perp Q(\mathcal{K}(H_6) \otimes B)e^\perp = \dot{\Theta}_2(A)^c$, this shows that $[\pi_6(u \oplus 1_3)]_{\dot{\Theta}_2(A)^c} = [\dot{z}]_{\dot{\Theta}_2(A)^c} = 0$. Note that $D_{\Theta_2} \cong \mathbf{M}_2(D_\Theta)$ and $\dot{\Theta}_2(A)^c \cong \mathbf{M}_2(\dot{\Theta}(A)^c)$, so that $[\dot{u}]_{\dot{\Theta}(A)^c} = 0$. Consequently, applying the K-theory exact sequence arising from (3.3), we may write $[u]_{K_1(D_\Theta)} = j_*(x)$ for some $x \in K_1(\mathcal{K}(H_3) \otimes B)$. ■

Notes 3.3.2 We are in fact using the Skandalis (nonunital) version of Paschke-Valette duality, (cf. [Pas81], [Val83], [Ska88], [Hig95]), in which the isomorphism

$$\mu_1 : KK(A, B) \to K_1(\dot{\Theta}(A)^c e)$$

is seen to map elements of the form $[\Theta, \Theta, u]$ to $[w]$, with e and w as in our proof above. Applying only the fact that this map is well-defined, we conclude in our setting that $[w]_{\dot{\Theta}(A)^c e} = 0$.

Obtaining inner automorphisms

Recall that θ is an admissible scalar representation and that Θ denotes $0_2 \oplus \theta : A \longrightarrow M(\mathcal{K}(H_3) \otimes B)$. We define, for any $m \in \mathbb{N}$, a C^*-algebra E_m by

$$E_m = \{m{\cdot}\Theta(a) \mid a \in A\} + \mathcal{K}(H_{3m}) \otimes B + \mathbb{C}1_{3m},$$

considered as a subalgebra of $M(\mathcal{K}(H_{3m}) \otimes B)$.

We need also to consider, for a unital C^*-algebra D, the group $\mathrm{Aut}_0(D)$ of automorphisms of D which are connected to the identity by a norm continuous path. It follows from [Ped79, 8.6.12,8.7.8] that whenever D is a unital separable C^*-algebra, $\mathrm{Aut}_0(D) \subseteq \overline{\mathrm{Inn}(D)}$.

Proposition 3.3.3 *Let A, B, φ, ψ and θ be as in Proposition 3.2.2. Whenever a finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$ is given, there exist $n \in \mathbb{N}$ and a unitary $V \in E_{n+1}$ such that*

$$\left\| V\left(\begin{bmatrix}\varphi(a) & & \\ & 0_1 & \\ & & \theta(a)\end{bmatrix} \oplus n{\cdot}\Theta(a)\right)V^* - \begin{bmatrix}\psi(a) & & \\ & 0_1 & \\ & & \theta(a)\end{bmatrix} \oplus n{\cdot}\Theta(a)\right\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof: Apply Proposition 3.2.2 to get u_t, and choose t_0 with $|1-t_0|<\varepsilon/3$ and such that

$$\left\| u_t \begin{bmatrix} \varphi(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} {u_t}^* - \begin{bmatrix} \psi(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} \right\| < \varepsilon/3 \tag{3.6}$$

for all $a \in \mathcal{F}$ and all $t > t_0$. Fix $t \in (t_0, 1)$. Because of Proposition 3.2.4, Proposition 3.3.1 applies to $[\Theta, \Theta, u_t]$. We can then choose a unitary

$$w \in \mathcal{U}\left(\begin{bmatrix} 0_1 & & \\ & \mathcal{K}(H_1)\otimes B & \\ & & 0_1 \end{bmatrix} + 1_3 \right)$$

such that $[u_t]_{K_1(D_\Theta)} = [w^*]_{K_1(D_\Theta)}$. Consequently, there exists $n \geq 1$ and a homotopy

$$u_t \oplus w \oplus 1_{3n-3} \overset{\omega_s}{\rightsquigarrow} 1_{3n+3}.$$

in $\mathcal{U}_{n+1}(D_\Theta)$. By our definition of D_Θ, $\omega_s E_{n+1} \omega_s^* = E_{n+1}$, so we can define a norm continuous family of automorphisms of E_{n+1} by $\alpha_s = \mathrm{Ad}_{\omega_s}$. Clearly $\alpha_1 = \mathrm{id}$, and since $\mathrm{Ad}_w \circ \Theta = \Theta$ because of the special form of w, we get that α_0 acts as $\mathrm{Ad}_{u_t} \oplus n \cdot \mathrm{id}$ on elements of the form

$$\begin{bmatrix} b & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} \oplus n \cdot \Theta(a).$$

Hence we have found $\alpha \in \mathrm{Aut}_0(E_{n+1})$ such that

$$\alpha\left(\begin{bmatrix} x & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} \oplus n \cdot \Theta(a) \right) = u_t \begin{bmatrix} x & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} {u_t}^* \oplus n \cdot \Theta(a) \tag{3.7}$$

for all $a \in A$ and all $x \in \mathcal{K}(H_1) \otimes B$. As noted above, classical results about automorphisms allow us to find $V \in E_{n+1}$ such that $\|Va'V^* - \alpha(a')\| < \varepsilon/3$ for all $a' \in \mathcal{F}'$ given by

$$\mathcal{F}' = \left\{ \begin{bmatrix} \varphi(a) & & \\ & 0_1 & \\ & & \theta(a) \end{bmatrix} \oplus (n-1) \cdot \Theta(a) \,\middle|\, a \in \mathcal{F} \right\}.$$

Since $\varphi(a) \in \mathcal{K}(H_1) \otimes B$, we get the desired estimate from (3.6) and (3.7). ■

As promised above we are now going to abandon our practice of not identifying C^*-algebras living on isomorphic Hilbert spaces. Except in the proof of the following result, we will hence drop the indices $H_1, H_2, \ldots$, and talk only of the separable infinite Hilbert space H. We are going to consider

$$E_\gamma = \left\{ \begin{bmatrix} 0 & & \\ & 0 & \\ & & \gamma(a) \end{bmatrix} \,\middle|\, a \in A \right\} + \mathbf{M}_3(\mathcal{K}(H) \otimes B) + \mathbb{C} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \subset M(\mathcal{K}(H_3) \otimes B),$$

where γ is an absorbing (unital) representation of A on $M(\mathcal{K}(H) \otimes B)$.

Theorem 3.3.4 *Let A be a unital, separable, nuclear C^*-algebra and let B be a σ-unital C^*-algebra. Assume that φ and ψ are two $*$-homomorphisms from A to $\mathcal{K}(H) \otimes B$ which satisfy $[\varphi]_{KK} = [\psi]_{KK}$ in $KK(A, B)$. Let $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ be a unital absorbing representation. Whenever a finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$ is given, there exists a unitary $V \in E_\gamma$ such that*

$$\left\| V \begin{bmatrix} \varphi(a) & & \\ & 0 & \\ & & \gamma(a) \end{bmatrix} V^* - \begin{bmatrix} \psi(a) & & \\ & 0 & \\ & & \gamma(a) \end{bmatrix} \right\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof: Apply Proposition 3.3.3 with some admissible scalar representation θ. By the fact that both $(n+1)\cdot\theta$ and γ are absorbing as seen above, we can choose a sequence of unitaries $u_m \in \mathcal{L}_B(H_{3n+2} \otimes B, H_2 \otimes B)$ implementing the equivalence $0_1 \oplus \theta \oplus n \cdot \Theta \sim 0_1 \oplus \gamma$. Write $W_m = (1 \oplus u_m)V(1 \oplus u_m^*)$. With m sufficiently large, we have

$$\left\| W_m \begin{bmatrix} \varphi(a) & & \\ & 0_1 & \\ & & \gamma(a) \end{bmatrix} W_m^* - \begin{bmatrix} \psi(a) & & \\ & 0_1 & \\ & & \gamma(a) \end{bmatrix} \right\| < \varepsilon,$$

and since, by (ii) of Definition 3.1.2, $\mathrm{Ad}_{1\oplus u_m}(E_{n+1}) = E_\gamma$, we have that $W_m \in E_\gamma$. This proves the claim. ■

Notes 3.3.5 The idea of proving results about the automorphisms of the C^*-algebra E_γ in order to obtain uniqueness results for morphisms originates with Lin ([Lin97]) in the case $\gamma = d_\iota$.

3.4 Stable uniqueness

The importance of the auxiliary C^*-algebra E_γ used in Theorem 3.3.4 above becomes apparent when one attempts to employ our quasidiagonality condition 3.1.5 to truncate the absorbing extensions in play to something more manageable in terms of classification.

Achieving stable uniqueness

We look at φ and ψ, two $*$-homomorphisms between unital C^*-algebras A to B. To fix notation, let 0 denote the zero operator in $M(\mathcal{K}(H) \otimes B)$ and let $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$ be a quasidiagonal representation. We consider a quasidiagonalization $(\gamma_n) : A \longrightarrow \mathbf{M}_{r_n}(B)$ by (e_n), where we may and shall assume that (e_n) has the additional property that

$$e_n\varphi(a)e_n = \varphi(a) \quad e_n\psi(a)e_n = \psi(a). \quad n \geq 1.$$

Theorem 3.4.1 *Let A be a unital, separable, nuclear C^*-algebra and let B be a unital C^*-algebra. Assume that there exists a quasidiagonal unital absorbing representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$, and let $(\gamma_n) : A \longrightarrow \mathbf{M}_{r_n}(B)$ be a quasidiagonalization of γ by (e_n) as above.*

Suppose that $\varphi, \psi : A \longrightarrow B$ are two $*$-homomorphisms with $[\varphi]_{KK} = [\psi]_{KK}$ in $KK(A, B)$, such that $\varphi(1)$ is unitarily equivalent to $\psi(1)$. Then for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exist an integer n and a unitary $u \in \mathcal{U}_{r_n+1}(B)$ satisfying

$$\left\| u \begin{bmatrix} \varphi(a) & \\ & \gamma_n(a) \end{bmatrix} u^* - \begin{bmatrix} \psi(a) & \\ & \gamma_n(a) \end{bmatrix} \right\| < \varepsilon$$

for all $a \in \mathcal{F}$. Moreover we may arrange that $u(\varphi(1) \oplus \gamma_n(1))u^* = \psi(1) \oplus \gamma_n(1)$.

Proof: After conjugating ψ by a unitary in B we may assume that $\varphi(1) = \psi(1)$. We are going to compress by $e_n' = e_n \oplus e_n \oplus e_n$ (which is a quasicentral sequence in E_γ), in the conclusion of Theorem 3.3.4. We have

$$e_n' \begin{bmatrix} \varphi(a) & & \\ & 0 & \\ & & \gamma(a) \end{bmatrix} e_n' = \begin{bmatrix} \varphi(a) & & \\ & 0 & \\ & & \gamma_n(a) \end{bmatrix}$$

and a similar equation for ψ. It is crucial to our argument that the unitary V provided by Theorem 3.3.4 satisfies $\|[V, e_n']\| \to 0$ because $V \in E_\gamma$. Therefore by perturbing $e_n' V e_n'$ to a unitary v within $\mathcal{U}_{3r_n}(B)$, for some large n, we obtain:

$$\left\| v \begin{bmatrix} \varphi(a) & & \\ & 0 & \\ & & \gamma_n(a) \end{bmatrix} v^* - \begin{bmatrix} \psi(a) & & \\ & 0 & \\ & & \gamma_n(a) \end{bmatrix} \right\| < \varepsilon$$

for all $a \in \mathcal{F}$. Consider the projection $e = \varphi(1) \oplus \gamma_n(1) = \psi(1) \oplus \gamma_n(1)$. After a small perturbation of v we may assume that $vev^* = e$. Then $w = eve$ is partial isometry in $\mathbf{M}_{r_n+1}(B)$ with $w^*w = ww^* = e$, and the unitary $u = w + 1_{r_n+1} - e \in \mathcal{U}_{r_n+1}(B)$ will satisfy the conclusion of the Theorem. ■

4 Improved uniqueness results

This section contain successive refinements of uniqueness results derived from the last result of the previous section.

4.1 Stable uniqueness with bounds

If one specializes to the case $\gamma = d_\iota$ in Theorem 3.4.1, one obtains a stable approximate unitary equivalence of the form

$$\left\| u \begin{bmatrix} \varphi(a) & \\ & n \cdot \iota(a) \end{bmatrix} u^* - \begin{bmatrix} \psi(a) & \\ & n \cdot \iota(a) \end{bmatrix} \right\| < \varepsilon.$$

To make such a result useful in our quest to classify C^*-algebras, we need to refine our uniqueness results to the effect of controlling the number n. More specifically, we need to know that these integers can be chosen uniformly with respect to the targets; i.e. only depending on the source algebra and, of course, the requirements on how closely the two morphisms are to agree after composition by the unitary.

We also need to strengthen the theorem to allow for maps which are not $*$-homomorphisms and only induce the same element locally in $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$, rather than in $KK(A, B)$. To achieve such results, we are going to work with products of C^*-algebras, and we are going to depend on the results in Appendix A.1 regarding their K-theory. If A satisfies the UCT, then if follows from (2.1) that $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$ is isomorphic to Rørdam's group $KL(A, B)$ [Rør95].

Bounded stable uniqueness for $*$-homomorphisms

Theorem 4.1.1 *Let A be a simple, unital, nuclear, separable C^*-algebra satisfying the UCT. Then for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ with the following property. For any admissible target B, any unital embedding $\iota : A \longrightarrow B$ and any pair of $*$-homomorphisms $\varphi, \psi : A \longrightarrow B$ such that $\varphi_* = \psi_*$ in $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$, and $\varphi(1)$ is unitarily equivalent to $\psi(1)$, there exists a unitary $u \in \mathcal{U}_{n+1}(B)$ such that*

$$\left\| u \begin{bmatrix} \varphi(a) & \\ & n \cdot \iota(a) \end{bmatrix} u^* - \begin{bmatrix} \psi(a) & \\ & n \cdot \iota(a) \end{bmatrix} \right\| < \varepsilon.$$

for all $a \in \mathcal{F}$. Moreover we may arrange that $u(\varphi(1) \oplus n\cdot 1)u^* = \psi(1) \oplus n\cdot 1$.

Proof: Suppose not and fix $\mathcal{F}$ and ε for which the theorem fails. Then for any i we choose an admissible target algebra B_i equipped with an embedding $\iota_i : A \longrightarrow B_i$, and φ_i, ψ_i $*$-homomorphisms with $\varphi_{i*} = \psi_{i*}$, and $\varphi_i(1)$ unitarily equivalent to $\psi_i(1)$, yet

$$\inf_{u\in\mathcal{U}_{i+1}(B)} \max_{a\in\mathcal{F}} \left\| u \begin{bmatrix} \varphi_i(a) & \\ & i\cdot\iota_i(a) \end{bmatrix} u^* - \begin{bmatrix} \psi_i(a) & \\ & i\cdot\iota_i(a) \end{bmatrix} \right\| \geq \varepsilon.$$

We define $\Phi, \Psi, I : A \longrightarrow \prod B_i$ in the obvious way, and compose with the canonical map to get $\dot{\Phi}, \dot{\Psi}, \dot{I} : A \longrightarrow \prod B_i / \sum B_i$. Since Φ and Ψ induce the same element of $\prod \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B_i))$ by construction, we get from Theorem A.1.6(ii) that $\Phi_* = \Psi_*$ in $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(\prod B_i))$. Then of course also $(\dot{\Phi})_* = (\dot{\Psi})_*$, and by Theorem A.1.6(iii) we get that $[\dot{\Phi}]_{KK} = [\dot{\Psi}]_{KK}$ in $KK(A, \prod B_i / \sum B_i)$.

Since $\dot{I} : A \longrightarrow \prod B_i / \sum B_i$ is a unital simple embedding, we conclude by Theorem 3.4.1 that there exist n and a unitary $w \in \mathcal{U}_{n+1}(\prod B_i / \sum B_i)$ intertwining $\dot{\Phi} \oplus n\cdot \dot{I}$ and $\dot{\Psi} \oplus n\cdot\dot{I}$ up to ε on $\mathcal{F}$. Let $u = (u_i) \in \mathcal{U}_{n+1}(\prod B_i)$ be a unitary lifting w. Then

$$\limsup_i \max_{a\in\mathcal{F}} \left\| u_i \begin{bmatrix} \varphi_i(a) & \\ & n\cdot\iota_i(a) \end{bmatrix} u_i^* - \begin{bmatrix} \psi_i(a) & \\ & n\cdot\iota_i(a) \end{bmatrix} \right\| < \varepsilon.$$

yielding a contradiction after projecting onto $\mathbf{M}_{n+1}(B_i)$ for large i. The last part of the proof is done exactly as the last part of the proof of Theorem 3.4.1. ■

Remark 4.1.2 If φ and φ are as in the conclusion of either Theorem 3.4.1 or Theorem 4.1.1, it follows immediately from the definition of K-theory that $\varphi_* = \psi_* : \underline{\mathbf{K}}(A) \to \underline{\mathbf{K}}(B)$.

Remark 4.1.3 Under assumptions restricting the algebraic complexity on $K_*(A)$ and $K_*(B)$ the result above can be simplified somewhat. If we add, for instance, the assumptions that $K_0(A)$ be torsion free and $K_0(B)$ be divisible, we need only require that $\varphi_* = \psi_*$ on $K_*(A)$. This is done basing the proof instead on injectivity of the maps

$$\mathrm{Hom}(K_*(A), K_*(\prod B_i)) \longrightarrow \prod \mathrm{Hom}(K_*(A), K_*(B_i))$$

$$KK(A, \prod B_i / \sum B_i) \longrightarrow \mathrm{Hom}(K_*(A), K_*(\prod B_i / \sum B_i)).$$

We get the latter by applying the UCT and note that since $K_0(A)$ is torsion free

$$\mathrm{Ext}\,(K_0(A), K_1(\prod B_i / \sum B_i)) = \mathrm{Pext}\,(K_0(A), K_1(\prod B_i / \sum B_i)) = 0$$

from Corollary A.1.5(i), and that

$$\mathrm{Ext}\,(K_1(A), K_0(\prod B_i / \sum B_i)) = 0$$

since, along the lines of the first half of Lemma A.1.4, if all $K_0(B_i)$ are divisible, then so is $K_0(\prod B_i)$.

Bounded stable uniqueness for approximate morphisms

We refer the reader to Appendix A.2 for a discussion of partially defined maps on $\underline{\mathbf{K}}(-)$ and a definition of $\underline{\mathbf{K}}$-triples.

Theorem 4.1.4 *Let A be a simple unital, nuclear, separable C^*-algebra satisfying the UCT. For any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exist $n \in \mathbb{N}$, and a $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any admissible target B, and any three completely positive contractions $\varphi, \psi, \tau : A \longrightarrow B$ which are δ-multiplicative on $\mathcal{G}$, with τ unital and $\varphi_\sharp(p) = \psi_\sharp(p)$ in $\underline{\mathbf{K}}(B)$ for all $p \in \mathcal{P}$, and such that $\varphi(1)$ and $\psi(1)$ are unitarily equivalent projections, there exists a unitary $u \in \mathcal{U}_{n+1}(B)$ such that*

$$\left\| u \begin{bmatrix} \varphi(a) & \\ & n\cdot\tau(a) \end{bmatrix} u^* - \begin{bmatrix} \psi(a) & \\ & n\cdot\tau(a) \end{bmatrix} \right\| < \varepsilon$$

for all $a \in \mathcal{F}$. One may arrange that $u(\varphi(1) \oplus n\cdot 1)u^* = \psi(1) \oplus n\cdot 1$.

Proof: Seeking a contradiction we suppose that there is $\mathcal{F}$ and ε such that with $n = n(A, \mathcal{F}, \varepsilon)$ provided by Theorem 4.1.1, no $\underline{\mathbf{K}}$-triple will work. We choose sequences of $\underline{\mathbf{K}}$-triples $(\mathcal{P}_i, \mathcal{G}_i, \delta_i)$ with the properties

(i) $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$, and $\bigcup_{i\in\mathbb{N}} \mathcal{P}_i$ exhausts the semigroup

$$\bigcup_{m\in\mathbb{N}} \mathrm{Proj}(A \otimes C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K})/\approx$$

(ii) $\mathcal{G}_i \subseteq \mathcal{G}_{i+1}$, $\overline{\bigcup_i \mathcal{G}_i} = A$.

(iii) $\delta_i > \delta_{i+1}$, $\delta_i \longrightarrow 0$.

By our assumption, we can then choose admissible targets B_i, which we may assume are of the same type, and φ_i, ψ_i and τ_i which are δ_i-multiplicative on $\mathcal{G}_i$ and satisfy $(\varphi_i)_\sharp(p) = (\psi_i)_\sharp(p)$ in $\underline{\mathbf{K}}(B)$ for all $p \in \mathcal{P}$, and $\varphi_i(1)$, $\psi_i(1)$ are unitarily equivalent projections; yet

$$\inf_{u\in\mathcal{U}_{n+1}(B)} \max_{a\in\mathcal{F}} \left\| u \begin{bmatrix} \varphi_i(a) & \\ & n\cdot\tau_i(a) \end{bmatrix} u^* - \begin{bmatrix} \psi_i(a) & \\ & n\cdot\tau_i(a) \end{bmatrix} \right\| \geq \varepsilon.$$

Define $\Phi, \Psi, T : A \longrightarrow \prod B_i$ from the sequences $(\varphi_n), (\psi_n)$ and (τ_n), and compose with the canonical map to get $\dot{\Phi}, \dot{\Psi}, \dot{T} : A \longrightarrow \prod B_i / \sum B_i$. These maps are in fact $*$-homomorphisms by (ii) and (iii) above, so $\dot{T}$ provides a unital simple embedding of A into $\prod B_i / \sum B_i$, and $\dot{\Phi}, \dot{\Psi}$ induce maps

$$\dot{\Phi}_*, \dot{\Psi}_* : \underline{\mathbf{K}}(A) \longrightarrow \underline{\mathbf{K}}\left(\prod B_i \Big/ \sum B_i\right).$$

We are going to show that $\dot{\Phi}_* = \dot{\Psi}_*$.

We may check this on $p \in \mathcal{P}_j$ by (i) above. Let $C = C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K}$, with m chosen appropriately. In the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_0\left((\sum B_i) \otimes C\right) \arrow[r] \arrow[d, "\cong"] & K_0\left((\prod B_i) \otimes C\right) \arrow[r] \arrow[d, "\eta"] & K_0\left(\frac{\prod B_i}{\sum B_i} \otimes C\right) \arrow[r] \arrow[d, "\dot{\eta}"] & 0 \\
0 \arrow[r] & \sum K_0\left(B_i \otimes C\right) \arrow[r] & \prod K_0\left(B_i \otimes C\right) \arrow[r, "\kappa"] & \frac{\prod K_0\left(B_i \otimes C\right)}{\sum K_0\left(B_i \otimes C\right)} \arrow[r] & 0
\end{tikzcd}$$

η is injective from Theorem A.1.6(i) because it can be naturally identified with a component of $\underline{\boldsymbol{\eta}}$. It is not hard to check that the above diagram has exact rows. (The first row is induced by a quasidiagonal extension). By the five-lemma, $\dot{\eta}$ is also injective, and hence it suffices to show that

$$\dot{\eta}[(\dot{\Phi} \otimes \mathrm{id}_C)(p)] = \dot{\eta}[(\dot{\Psi} \otimes \mathrm{id}_C)(p)].$$

With χ_0 as in Appendix A.2, lift $(\dot{\Phi} \otimes \mathrm{id})(p)$ first to a self-adjoint element $(\Phi \otimes \mathrm{id})(p)$ in $(\prod B_i) \otimes C$, and then to a projection q in

$$\chi_0((\Phi \otimes \mathrm{id})(p)) + \left(\sum B_i\right) \otimes C \subseteq \left(\prod B_i\right) \otimes C.$$

Then $\eta([q]) = ([q_i])$ where $q_i = \chi_0((\varphi_i \otimes \mathrm{id})(p))$ for all i larger than some i_Φ, since η is induced by a family of $*$-homomorphisms. We conclude that

$$\dot{\eta}((\dot{\Phi} \otimes \mathrm{id})_*(p)) = \kappa\eta([q]) = [\chi_0((\varphi_i \otimes \mathrm{id})(p))]_{i \geq i_\Phi} + \sum K_0(B_i \otimes C).$$

Similarly,

$$\dot{\eta}((\dot{\Psi} \otimes \mathrm{id})_*(p)) = [\chi_0((\psi_i \otimes \mathrm{id})(p))]_{i \geq i_\Psi} + \sum K_0(B_i \otimes C),$$

and these elements agree since the sequences coincide for $i \geq i_\Phi, i_\Psi, j$.

Having proved that $\dot{\Phi}_* = \dot{\Psi}_*$, and since $\dot{\Phi}(1)$ is unitarily equivalent to $\dot{\Psi}(1)$, we may apply Theorem 4.1.1 to find a unitary $w \in \mathcal{U}_{n+1}(\prod B_i / \sum B_i)$ such that

$$\left\| w \begin{bmatrix} \dot{\Phi}(a) & \\ & n \cdot \dot{T}(a) \end{bmatrix} w^* - \begin{bmatrix} \dot{\Psi}(a) & \\ & n \cdot \dot{T}(a) \end{bmatrix} \right\| < \varepsilon.$$

for all $a \in \mathcal{F}$. Note that we may apply Theorem 4.1.1 since whenever B_i is a sequence of admissible targets, $\prod B_i / \sum B_i$ is an admissible target by Theorem A.1.6(iv). We finish the argument by lifting the unitary as above. ∎

Remark 4.1.5 As in Remark 4.1.3, the premises of Theorem 4.1.4 simplify under extra assumptions on $K_*(A)$ and $K_*(B)$. If $K_0(A)$ is torsion free and $K_0(B)$ is divisible, one needs only check that φ and ψ agree on a K_*-triple. This is because divisibility passes from $K_0(B_i)$ to $K_0(\prod B_i / \sum B_i)$ as outlined in Remark 4.1.3.

5 Existence

In this section we show how to realize locally a given KK-element by a difference of completely positive contractions (see Theorem 5.1.5).

5.1 Realizing group homomorphisms

We refer the reader to Appendix A.2 for the definition of $\underline{\mathbf{K}}$-triples and of the map χ_0.

Lemma 5.1.1 *Let E_i and F_i be projections in $M(\mathcal{K}(H)\otimes B)$ with*

$$\|E_0 - E_1\| < \tfrac{1}{3}, \quad \|F_0 - F_1\| < \tfrac{1}{3}, \quad E_i - F_i \in \mathcal{K}(H)\otimes B$$

for $i \in \{0,1\}$. Then $[E_0, F_0] = [E_1, F_1]$ in $KK_h(\mathbb{C}, B) \cong KK(\mathbb{C}, B)$.

Proof: If $X_t = (1-t)E_0 + tE_1$ and $Y_t = (1-t)F_0 + tF_1$, then $\mathrm{sp}(X_t), \mathrm{sp}(Y_t) \subseteq [0, 1/3] \cup [2/3, 1]$ for all $t \in [0,1]$. Define $E_t = \chi_0(X_t)$ and $F_t = \chi_0(Y_t)$. Then (E_t, F_t), $0 \le t \le 1$ is a homotopy of Cuntz pairs from (E_0, F_0) to (E_1, F_1). ■

Lemma 5.1.2 *There exists $\lambda > 0$ such that whenever E, F are projections in $M(\mathcal{K}(H)\otimes B)$ with $E - F \in \mathcal{K}(H)\otimes B$ (i.e. $[E,F]$ defines a cycle in $KK_h(\mathbb{C}, B)$), and e is a projection in $\mathcal{K}(H)\otimes B$ with*

$$\|[e,E]\| < \lambda \qquad \|[e,F]\| < \lambda \qquad \left\|e^\perp (E-F) e^\perp\right\| < \lambda,$$

then the natural isomorphism between $KK_h(\mathbb{C}, B)$ and $K_0(B)$ takes $[E,F]$ to

$$[\chi_0(eEe)] - [\chi_0(eFe)].$$

Proof: Let $g = \chi_0(eEe)$, $g' = \chi_0(e^\perp E e^\perp)$, $h = \chi_0(eFe)$, $h' = \chi_0(e^\perp F e^\perp)$. For small λ, g, g', h, h' are all projections and we may apply Lemma 5.1.1 to get

$$[E,F] = [g \oplus g', h \oplus h'] = [g,h] + [g', h'] = [g,h] + [g', g'] = [g,h]$$

in $KK_h(A,B)$. Since $g, h \in \mathcal{K}(H)\otimes B$, the isomorphism between $KK_h(\mathbb{C}, B)$ and $K_0(B)$ takes $[g,h]$ to $[g] - [h]$. ■

Lemma 5.1.3 *Let A be a unital nuclear C^*-algebra, and let $\varepsilon > 0$ and a finite set $\mathcal{F} \subseteq A$ be given. Then there is $\delta > 0$ such that for any unital C^*-algebra B and any completely positive contraction $\varphi : A \longrightarrow B$ which satisfies $\|\varphi(1)^2 - \varphi(1)\| < \delta$, then there exists a completely positive contraction ψ with $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$ such that $\varphi(1)$ is a projection.*

Proof: Seeking a contradiction, we suppose that there exist $\mathcal{F}$, ε as well as sequences of unital C^*-algebras B_i and completely positive contractions $\varphi_i : A \longrightarrow B_i$ with

$$\left\|\varphi_i(1)^2 - \varphi_i(1)\right\| \longrightarrow 0,$$

yet for all completely positive contractions $\psi : A \longrightarrow B_i$ with $\psi(1)$ a projection, we have

$$\sup_{a\in\mathcal{F}} \|\varphi_i(a) - \psi(a)\| \geq \varepsilon.$$

If we let $\Phi = (\varphi_i) : A \longrightarrow \prod B_i$ and $\dot{\Phi} : A \longrightarrow \prod B_i / \sum B_i$, then $\dot{\Phi}(1) = E$ is a projection. Let (e_i) be a projection of $\prod B_i$ lifting E. This gives an isomorphism

$$\gamma : \frac{\prod e_i B_i e_i}{\sum e_i B_i e_i} \longrightarrow E \frac{\prod B_i}{\sum B_i} E,$$

and by the unital version of the Choi-Effros theorem [CE76] we can lift $\gamma^{-1}\dot{\Phi}$ to a unital completely positive map $\Psi = (\psi_i) : A \longrightarrow \prod e_i B_i e_i$. Then $\psi_i(1) = e_i$, and $\|\psi_i(a) - \varphi(a)\|$ tends to zero, leading to a contradiction. ■

We now come to the main results in this section. The reader is referred to Remark 3.1.6 for examples of quasidiagonal absorbing representations.

Any element $\alpha \in KK(A, B)$ induces a morphism $\alpha_* : \underline{\mathbf{K}}(A) \to \underline{\mathbf{K}}(B)$. If p is a projection in A, and α is given by a Cuntz pair (τ, γ), then α_* takes $[p] \in K_0(A) \cong KK_h(\mathbb{C}, A)$ to $[\tau(p), \gamma(p)] \in KK_h(\mathbb{C}, B)$.

Theorem 5.1.4 *Let A be a unital, separable, nuclear C^*-algebra and let B be a unital C^*-algebra. Assume that there exists a quasidiagonal unital absorbing representation $\gamma : A \longrightarrow M(\mathcal{K}(H) \otimes B)$, and let $(\gamma_n) : A \longrightarrow \mathbf{M}_{r_n}(B)$ be a quasidiagonalization of γ by (e_n) as in Definition 3.1.5.*

For any $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{F}, \delta)$ there exist N and a completely positive contraction

$$\sigma : A \longrightarrow \mathbf{M}_{2r_N}(B)$$

such that σ and γ_N are both δ-multiplicative on $\mathcal{F}$ and satisfy

$$\sigma_\sharp(p) - (\gamma_N)_\sharp(p) = \alpha_*[p]$$

for all $p \in \mathcal{P}$.

Proof: For any admissible scalar representation θ, if $\Theta = 0 \oplus \theta$, then by [Ska88, 2.6] α is represented by a KK-cycle (Θ, Θ, x). By using the standard simplification given by Proposition 17.4.3 in [Bla86] we may assume that x is a contraction. Finally we may replace x by the unitary

$$u = \begin{bmatrix} x & (1-xx^*)^{1/2} \\ -(1-x^*x)^{1/2} & x^* \end{bmatrix}$$

Hence, in $KK(A, B)$ we have $\alpha = [\Theta, \Theta, u] = [u\Theta u^*, \Theta, 1]$. This shows that for any admissible scalar representation θ, α is represented by some Cuntz pair $(\rho, \Theta) : A \longrightarrow M(\mathcal{K}(H) \otimes B)$. We have

$$\rho(a) - \Theta(a) \in \mathcal{K}(H) \otimes B$$

for all $a \in A$.

Since γ is absorbing, we have $\theta \sim \gamma$ in the sense of Definition 3.1.2. Dilating the unitaries trivially, we get a sequence $u_i \in \mathcal{U}(M(\mathcal{K}(H) \otimes B))$ with

$$\Gamma(a) - u_i\Theta(a)u_i^* \in \mathcal{K}(H) \otimes B \qquad \|\Gamma(a) - u_i\Theta(a)u_i^*\| \longrightarrow 0, \tag{5.1}$$

where $\Gamma = 0 \oplus \gamma$. Note that for each i, $(u_i\rho u_i^*, u_i\Theta u_i^*)$ is a Cuntz pair representing α. Then $X_i = (u_i\rho u_i^*, \Gamma)$ is also a Cuntz pair, and $[X_i]_* : \underline{\mathbf{K}}(A) \longrightarrow \underline{\mathbf{K}}(B)$ converges pointwise to the group homomorphism induced by α as a consequence of Lemma 5.1.1. Fix i large enough that $[X_i]_*[p] = \alpha_*[p]$ for each $p \in \mathcal{P}$.

Let $\tau = u_i\rho u_i^*$ and let $e_n \in \mathbf{M}_{r_n}(B)$ be as in the statement. Then $f_n = e_n \oplus e_n \in matr M_{2r_n}(B)$ is an approximate unit of projections which in the obvious sense quasidiagonalizes Γ into $\gamma_n : A \longrightarrow \mathbf{M}_{2r_n}(B)$. We set $\sigma_n(a) = f_n\tau(a)f_n$ and note that

$$[f_n, \tau(a)] \longrightarrow 0 \qquad f_n^\perp(\tau(a) - \Gamma(a))f_n^\perp \longrightarrow 0$$

for $a \in A$, since $\tau(a) - \Gamma(a) \in \mathcal{K}(H) \otimes B$. We have that $\mathcal{P} \subseteq C$ where

$$C = \bigoplus_{m \le M} A \otimes C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K}.$$

Therefore for N large enough, σ_N is δ-multiplicative on $\mathcal{F}$ and

$$\|[f_N \otimes 1_C, (\tau \otimes \mathrm{id}_C)(p)]\| < \lambda \qquad \|[f_N \otimes 1_C, (\Gamma \otimes \mathrm{id}_C)(p)]\| < \lambda$$
$$\left\| f_N^\perp \otimes 1_C((\tau \otimes \mathrm{id}_C)(p) - (\Gamma \otimes \mathrm{id}_C)(p)) f_N^\perp \otimes 1_C \right\| < \lambda,$$

for all $p \in \mathcal{P}$. By A.2 and Lemma 5.1.2 we have

$$\begin{aligned}
&(\sigma_N)_\sharp(p) - (\Gamma_N)_\sharp(p) = \\
&\quad [\chi_0(f_N \otimes 1_C(\tau \otimes \mathrm{id}_C)(p) f_N \otimes 1_C)] - [\chi_0(f_N \otimes 1_C(\Gamma \otimes \mathrm{id}_C)(p) f_N \otimes 1_C)] = \\
&\quad [(\tau \otimes \mathrm{id}_C)(p), (\Gamma \otimes \mathrm{id}_C)(p)]_{KK_h} = [X_i]_*[p] = \alpha_*[p]
\end{aligned}$$

∎

Next we specialize the existence result to the case of a quasidiagonal source A. An application for purely infinite C^*-algebras can be found in Theorem 6.3.3.

A C^*-algebra is RFD or *residually finite-dimensional* if it has a separating family of finite-dimensional representations. We say that A is *locally RFD* if for any finite set $\mathcal{F}$ and any $\varepsilon > 0$, there exists an RFD subalgebra A' of A, containing all elements of $\mathcal{F}$ up to ε.

Theorem 5.1.5 *Let A, B be C^*-algebras with A nuclear, unital and quasidiagonal and B unital, and let $\alpha \in KK(A, B)$. For any $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{F}, \delta)$ there exist N and completely positive contractions*

$$\sigma : A \longrightarrow \mathbf{M}_N(B) \qquad \mu : A \longrightarrow \mathbf{M}_N(\mathbb{C}1_B)$$

which are δ-multiplicative on $\mathcal{F}$ and satisfy

$$\sigma_\sharp(p) - \mu_\sharp(p) = \alpha_*[p]$$

for all $p \in \mathcal{P}$. We may arrange that $\sigma(1)$ and $\mu(1)$ are both projections. Moreover if A is locally RFD and if $\varepsilon > 0$ is given, we can arrange that there is a unital RFD subalgebra D of A such that $\mathcal{F} \subseteq_\varepsilon D$ and the restriction of μ to D is a $*$-homomorphism.

Proof: Since A is quasidiagonal, it has a quasidiagonal admissible scalar representation θ. Note that, using projections $e_n \in \mathcal{K}(H) \otimes 1_B$ we obtain a quasidiagonalization consisting of maps $\theta_n : A \longrightarrow \mathbf{M}_{r_n}(\mathbb{C}1_B)$. The first claim follows from Theorem 5.1.4, by taking $\mu = \theta_n$ for some large n.

Combining Lemma 5.1.3 and Lemma A.2.7 we may replace σ and μ with completely positive contractions which map the unit of A to a projection. In case A is locally RFD, we find first an unital RFD subalgebra D of A such that $F \subseteq_{\varepsilon_1} D$. Then we work with an admissible representation θ whose restriction to D is a direct sum of finite dimensional representations. It is then clear that one can choose the quasidiagonalization such that the restriction of θ_n to D is a $*$-homomorphism for all n. ■

Notes 5.1.6 An existence result for locally RFD C^*-algebras has been independently obtained by Lin [Lin98a]. Our result is more general and moreover it applies to the purely infinite C^*-algebras. Nevertheless, the premises of the two results are the same in the simple case, since it is proved in [BK97] that every simple, nuclear and quasidiagonal C^*-algebra is locally RFD,

6 Classification

In this section, we present applications of the uniqueness and existence results to classification problems. The first part is devoted to a class of finite C^*-algebras which allow a further refinement of the uniqueness result, leading to a complete classification of C^*-algebras in this class having K_0-group $\mathbb{Q}$. In the last part, we show how the results also apply to reprove the classification theorem for purely infinite C^*-algebras by Kirchberg and Phillips.

6.1 Approximate unitary equivalence

In a class of C^*-algebras studied by H. Lin it is possible to absorb the stabilization required in Theorem 4.1.4, leading to further improved uniqueness results. Furthermore, this class is contained in our class of admissible targets. In this section, we develop these points.

Definition 6.1.1 ([Lin98b]) A simple unital C^*-algebra is called TAF (tracially approximately finite dimensional) if for any finite subset $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and any nonzero projection $q \in A$ there is a projection $p \in A$, $p \neq 1_A$, and there is a finite dimensional C^*-algebra $C \subseteq p^\perp A p^\perp$ with $p^\perp \in C$ such that

(i) $\|[p, a]\| < \varepsilon$ for all $a \in \mathcal{F}$.
(ii) $\operatorname{dist}(p^\perp a p^\perp, C) < \varepsilon$ for all $a \in \mathcal{F}$.
(iii) $upu^* \leq q$ for some unitary $u \in A$.

Example 6.1.2 *The real rank zero approximately (sub)homogeneous C^*-algebras classified in [EG96] and [DG97] are all TAF algebras. Also the class of examples of nonnuclear subalgebras of AF algebras constructed in [Dar97] consists entirely of TAF algebras.*

The conditions (i)–(ii) from [Pop97] imply quasidiagonality of A. Let us summarize a few other structural results on TAF C^*-algebras that we shall need. Lemmas 6.1.3 and 6.1.5 are from [Lin98b].

Lemma 6.1.3 *([Lin98b, 3]) Let A be a simple unital TAF C^*-algebra. Then A has real rank zero, stable rank one, and $K_0(A)$ is weakly unperforated in the sense of [Bla86]. When $p \in A$ is a projection and $n \in \mathbb{N}$, then both pAp and $\mathbf{M}_n(A)$ are simple unital TAF C^*-algebras.*

Lemma 6.1.4 *Let B be an infinite dimensional unital separable simple C^*-algebra of real rank zero, stable rank one, with $K_0(B)$ weakly unperforated. Then for any $n \geq 1$ and any nonzero projection $f \in B$ there are mutually orthogonal projections $e^1, \dots, e^n$ and r in B such that $e^1 + \cdots + e^n + r = 1_B$ with $[e^1] = \cdots = [e^n]$ and $[r] < [f]$.*

Proof: Let $QT(B)$ denote the normalized quasitraces on B. The image of the natural map $\rho_0 : K_0(B) \longrightarrow \mathrm{Aff}(QT(B))$ is uniformly dense by [Bla86, 6.9.3] and $K_0 B$ has the strict ordering induced from ρ_0 by [Bla86, 6.9.2]. If e is a projection we write $\hat{e} = \rho_0(e)$. By simplicity, we find N big enough such that $N[f] > [1]$. If $\varepsilon = 1/nN$, then $1/n - \varepsilon > 0$ and $n\varepsilon\,\hat{1} < \hat{f}$. Since the image of ρ_0 is uniformly dense, there is a projection $e \in B$ such that $(1/n - \varepsilon)\hat{1} < \hat{e} < 1/n\,\hat{1}$. Therefore $0 < 1 - n\hat{e} < n\varepsilon\hat{1} < \hat{f}$, hence

$$0 < [1] - n[e] < [f].$$

This is readily seen to imply the statement. Indeed if $d^i \in B$ are projections equivalent to e then $d^1 \oplus \cdots \oplus d^n$ is equivalent to a subprojection d of 1_B. If $r = 1 - d$, then $[r] = [1] - n[e] < [f]$. ■

The TAF C^*-algebras are prone to classification because of the following factorization property.

Lemma 6.1.5 *([Lin98b]) Let A be a simple unital TAF C^*-algebra. For any $n \geq 1$, any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there are projections $p, q \in A$ with $p^\perp A p^\perp \cong \mathbf{M}_n(qAq)$ and $[p] \leq [q]$ and such that there exists an approximate factorization of id_A*

$$\begin{tikzcd} A \arrow[rr, equal, "\mathrm{id}_A"] \arrow[dr, "\nu"'] & & A \\ & pAp \oplus \mathbf{M}_n(C) \arrow[ur, hook, "\mu"'] & \end{tikzcd}$$

with $\|\mu\nu(a) - a\| < \varepsilon$ for $a \in \mathcal{F}$, where C is a unital finite dimensional C^*-subalgebra of qAq, $\nu(a) = pap \oplus (\eta(a) \otimes 1_n)$ is ε-multiplicative on $\mathcal{F}$ with $\eta : A \longrightarrow C$ a unital completely positive contractive map, and μ is a unital $*$-monomorphism whose restriction to pAp is the natural inclusion.

Proof: We include a proof of this result which is somewhat different from the original proof of Lin. We don't require A to be nuclear. We may assume that A is infinite dimensional; it suffices to prove the statement with 8ε instead of ε. Since A is TAF we find a projection $P \in A$ and a finite dimensional C^*-algebra C with $P^\perp \in C \subset P^\perp A P^\perp$ such that for all $a \in \mathcal{F}$ we have

(i) $\|[P,a]\| < \varepsilon$
(ii) $P^\perp a P^\perp \in_\varepsilon C$
(iii) $(n+2)[P] \leq [1]$.

The idea of the proof is to find a unital embedding of $\mathbf{M}_n(\mathbb{C}) \oplus \mathbb{C}$ into the relative commutant of C in $P^\perp A P^\perp$ such that the image of $\mathbb{C}$ is supported by a very small projection. Write $C \cong \mathbf{M}_{m(1)} \oplus ... \oplus \mathbf{M}_{m(k)}$ and let $e_1, ..., e_k$ be the minimal central projections of C. Let B_i be the relative commutant of $e_i C e_i \cong \mathbf{M}_{m(i)}$ in $e_i A e_i$. Then $e_i A e_i \cong \mathbf{M}_{m(i)}(B_i)$ hence B_i is TAF being isomorphic to a corner of A. Let f be a nonzero projection in A with $(n+1)k[f] < [P]$. For each $1 \leq i \leq k$ we apply Lemma 6.1.4 for B_i. We obtain:

$$e_i = e_i^1 + \cdots + e_i^n + r_i$$

where $e_i^j, 1 \leq j \leq n$ are mutually equivalent projections in B_i and $[r_i] < [f]$ in $K_0(A)$. Set $e^j = e_1^j + \cdots + e_k^j$, $1 \leq j \leq n$, $r = r_1 + \cdots + r_k$ and $e = e^1 + \cdots + e^n$. Note that $P^\perp = e^1 + \cdots + e^n + r$ with e^j mutually equivalent in the relative commutant of C in $P^\perp A P^\perp$. We have $(n+1)[r] = (n+1)([r_1] + \cdots + [r_k]) < (n+1)k[f] < [P]$. Therefore

$$(n+1)([P] + [r]) < (n+2)[P] \leq [1] = [P] + n[e^1] + [r]$$

hence $n([P] + [r]) < n[e^1]$. By weak unperforation we get $[P + r] \leq [e^1]$. We are now ready to complete the proof. By Arveson's extension theorem, the inclusion $C \hookrightarrow A$ extends to a completely positive contraction $E : A \to C$. We have $E(x) = x$ for $x \in C$ hence $\|a - E(a)\| \leq 2\,\mathrm{dist}(a, C)$ for $a \in A$. Using (i) and (ii) we have for $a \in \mathcal{F}$

$$\begin{aligned} a &\sim_{2\varepsilon} PaP + P^\perp a P^\perp \sim_{2\varepsilon} PaP + E(P^\perp a P^\perp) \\ &= PaP + E(P^\perp a P^\perp) r + E(P^\perp a P^\perp) e \sim_{4\varepsilon} (P+r)a(P+r) + E(P^\perp a P^\perp) e. \end{aligned}$$

The last estimate follows by compressing the estimate $a \sim_{4\varepsilon} PaP + E(P^\perp a P^\perp)$ by $P + r$. It follows that $a \sim_{8\varepsilon} (P+r)a(P+r) + E(P^\perp a P^\perp) e$. We finish the proof by setting $p = P + r$, $q = e^1$ and noting that $E(P^\perp a P^\perp) e$ is of the form $\eta(a) \otimes 1_n$ since $e = e^1 + \cdots + e^n$ with e^j mutually equivalent in the relative commutant of C in $P^\perp A P^\perp$. ■

The uniqueness Theorems 4.1.1 and 4.1.4 apply to TAF algebras because of the following Proposition.

Proposition 6.1.6 *A simple unital infinite-dimensional TAF C^*-algebra is an admissible target algebra (of finite type).*

Proof: Let B be a simple unital TAF C^*-algebra. We get (i) and (ii) of Definition 2.1.3 by two results of Rieffel ([Rie83], [Rie87]). For (iii.1), note that if $nx \geq 0$ then $nx + n[1_B] > 0$ and $x + [1_B] > 0$ by weak unperforation. Finally, to prove (iv.1), assume that $\dim(B) = \infty$ and let x and n be given. The image of the natural map $\rho_0 : K_0(B) \longrightarrow \mathrm{Aff}(QT(B))$ is uniformly dense by [Bla86, 6.9.3], so we can find $z \in K_0(B)$ with $\rho_0(x) - 1 < \rho_0(nz) < \rho_0(x) + 1$. By [Bla86, 6.9.2] we have $x - [1_B] \leq nz \leq x + [1_B]$ so $y = x - nz$ will work. ■

Theorem 6.1.7 *Let A be a simple unital, nuclear, separable TAF C^*-algebra satisfying the UCT. Then for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exists a $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with*

the following property. For any unital simple infinite-dimensional TAF C^*-algebra B, and any two unital completely positive contractions $\varphi, \psi : A \longrightarrow B$ which are δ-multiplicative on $\mathcal{G}$, with $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}$, there exists a unitary $u \in \mathcal{U}(B)$ such that

$$\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof: Let us begin by outlining the proof. We first construct such a unitary in the special case where φ, ψ are $*$-homomorphisms agreeing on all of $\underline{\mathbf{K}}(A)$. This involves invoking Lin's factorization result 6.1.5 to pass to another pair of $*$-homomorphisms $\overline{\varphi}, \overline{\psi}$ which are on a special form. Because Lin's result only gives an approximate factorization, even though we start out with $*$-homomorphisms, our proof will take us to a setting where our uniqueness theorem for completely positive contractions Theorem 4.1.4 is needed. The general case will follow in the same way that Theorem 4.1.1 implies Theorem 4.1.4, by letting $n = 0$ and $T = 0$ in the proof of Theorem 4.1.4. We include a sketch for the benefit of the suspicious reader.

PART 1: Given $\mathcal{F}$ and ε, we are going to prove that whenever

- (i) $B = \prod B_i / \sum B_i$ with each B_i a unital simple infinite-dimensional TAF C^*-algebra (or B itself a unital simple infinite-dimensional TAF C^*-algebra).
- (ii) $\varphi, \psi : A \longrightarrow B$ are unital $*$-homomorphisms
- (iii) $\varphi_* = \psi_* : \underline{\mathbf{K}}(A) \longrightarrow \underline{\mathbf{K}}(B)$

then there exists $u \in \mathcal{U}(B)$ with $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

Let us thus fix n, $\mathcal{P}$, $\mathcal{G}$ and δ by applying Theorem 4.1.4 to $\mathcal{F}$ and $\varepsilon/3$. Furthermore, let p, q, C, ν and μ be given by Lemma 6.1.5 such that ν is δ-multiplicative on $\mathcal{G}$ and $\|\mu\nu(a) - a\| < \varepsilon/3$ for all $a \in \mathcal{F}$.

STEP 1A: Since B has stable rank one by Lemma 6.1.3, it has cancellation of projections, and because $\varphi_* = \psi_*$ we may assume, after conjugating ψ by a unitary in B, that the restrictions of φ and ψ to the finite dimensional algebra $\mu(\mathbb{C}p \oplus \mathbf{M}_n(\mathbb{C}1_C))$ are equal. Applying μ to the matrix units of $\mathbf{M}_n(\mathbb{C}1_C)$ we can define matrix units (q_{ij}) in A, where $q = q_{11} = 1_C$. Let

$$e = \varphi(p) = \psi(p) \qquad f_{ij} = \varphi(q_{ij}) = \psi(q_{ij}), \tag{6.1}$$

abbreviating $f = f_{11}$. Invoking cancellation again, since $[p] \le [q]$, we can find a projection e_0 and a unitary v in B with $v(e + e_0)v^* = f$. Let $g = e_0 \oplus 1_B \in \mathbf{M}_2(B)$ and note that $[g] = (n+1)\cdot[f]$. Hence an isomorphism $\gamma : g\mathbf{M}_2(B)g \longrightarrow \mathbf{M}_{n+1}(fBf)$ can be found. Denoting the matrix units of $\mathbf{M}_{n+1}(\mathbb{C}f)$ by $\tilde{f}_{ij}$ with $0 \le i, j \le n$ we may choose γ such that

$$\gamma\left(\begin{bmatrix} e_0b_0e_0 & \\ & ebe \end{bmatrix}\right) = \tilde{f}_{00}v(ebe + e_0b_0e_0)v^*\tilde{f}_{00} \qquad \gamma\left(\begin{bmatrix} 0 & \\ & f_{ij} \end{bmatrix}\right) = \tilde{f}_{ij} \tag{6.2}$$

Combining all of this, we get a $*$-homomorphism $\overline{\varphi}$ fitting in a diagram

$$\begin{tikzcd}
A \arrow[r, equal, "\mathrm{id}_A"] \arrow[dr, "\nu"'] & A \arrow[r, "\varphi"] & B \arrow[r, hook, "\iota_2"] & g\mathbf{M}_2(B)g \arrow[d, "\gamma"] \\
 & pAp \oplus \mathbf{M}_n(C) \arrow[u, hook, "\mu"] \arrow[rr, "\overline{\varphi}"'] & & \mathbf{M}_{n+1}(fBf),
\end{tikzcd}$$

where ι_2 sends B into the $(2,2)$ corner of $g\mathbf{M}_2(B)g$. Identifying $\overline{\varphi}$ using (6.1) and (6.2) we get that for all $d \in pAp$, $x \in \mathbf{M}_n(C)$,

$$\overline{\varphi}(d \oplus x) = \varphi''(d) \oplus (\varphi' \otimes \mathrm{id}_n)(x)$$

where

$$\varphi' : C \longrightarrow fBf \qquad \varphi'' : pAp \longrightarrow fBf$$

are defined as corestrictions of φ and $\mathrm{Ad}_v\,\varphi$, respectively. Furthermore, by symmetry of (6.1), the same procedure shows that $\overline{\psi} = \gamma\iota_2\psi\mu : pAp \oplus \mathbf{M}_n(C) \longrightarrow \mathbf{M}_{n+1}(fBf)$ has the form

$$\overline{\psi}(d \oplus x) = \psi''(d) \oplus (\psi' \otimes \mathrm{id}_n)(x)$$

for all $d \in pAp$, $x \in \mathbf{M}_n(C)$.

Step 1b: With $\iota_f : fBf \longrightarrow B$ we clearly have that $(\iota_f\varphi')_* = (\iota_f\psi')_*$ and $(\iota_f\varphi'')_* = (\iota_f\psi'')_*$. But since f is full in B as the image of a full projection under a unital map, we get by [Bro77] that ι_f induces an isomorphism from $\underline{\mathbf{K}}\,(fBf)$ to $\underline{\mathbf{K}}\,(B)$, so that $(\varphi')_* = (\psi')_*$ and $(\varphi'')_* = (\psi'')_*$. From this we may assume, after conjugating ψ by a unitary of fBf, that φ' and ψ' agree on C. Thus the maps $\overline{\varphi}\nu$ and $\overline{\psi}\nu$ are of the form

$$\overline{\varphi}\nu(a) = \begin{bmatrix} \varphi''\omega(a) & & & \\ & \varphi'\eta(a) & & \\ & & \ddots & \\ & & & \varphi'\eta(a) \end{bmatrix} \qquad \overline{\psi}\nu(a) = \begin{bmatrix} \psi''\omega(a) & & & \\ & \varphi'\eta(a) & & \\ & & \ddots & \\ & & & \varphi'\eta(a) \end{bmatrix}$$

where we define $\omega : A \longrightarrow pAp$ by $\omega(a) = pap$. Note that ω is δ-multiplicative on $\mathcal{G}$. Now $\varphi''\omega$, $\psi''\omega$ and $\varphi'\eta$ are δ-multiplicative on $\mathcal{G}$ and by Lemma A.2.4 we have that $(\varphi''\omega)_\sharp(p) = (\psi''\omega)_\sharp(p)$ for all $p \in \mathcal{P}$. Therefore, Theorem 4.1.4 applies to the triple of maps $(\varphi''\omega, \psi''\omega, \varphi'\eta)$ if we can prove that fBf is an admissible target. When B itself is a TAF C^*-algebra, so is fBf by Lemma 6.1.3, and Proposition 6.1.6 applies. When $B = \prod B_i / \sum B_i$ we note that there are projections $f_i \in B_i$ such that fBf is isomorphic to

$$\prod f_iB_if_i \Big/ \sum f_iB_if_i$$

and hence it is admissible of finite type by Lemma A.1.3 since all the $f_iB_if_i$ are. Thus, by Theorem 4.1.4, there is a partial isometry $v \in \mathbf{M}_{n+1}(fBf)$ such that $v^*v = \overline{\varphi}\nu(1)$, $vv^* = \overline{\psi}\nu(1)$ and and $\left\|v\overline{\varphi}\nu(a)v^* - \overline{\psi}\nu(a)\right\| < \varepsilon/3$ for all $a \in \mathcal{F}$. Since $\overline{\varphi}\nu(1) = \overline{\psi}\nu(1) = \gamma\iota_2(1)$ and $\gamma\iota_2(B) = \gamma\iota_2(1)\mathbf{M}_{n+1}(fBf)\gamma\iota_2(1)$, we have that $v = \gamma\iota_2(u)$ for some unitary $u \in B$. Since $\gamma\iota_2$ is isometric we obtain $\|u\varphi\mu\nu(a)u^* - \psi\mu\nu(a)\| < \varepsilon/3$, hence $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$, since $\|\mu\nu(a) - a\| < \varepsilon/3$ for all $a \in \mathcal{F}$.

Part 2: As the argument reducing to the case covered in Part 1 closely parallels that in the proof of Theorem 4.1.4, we only sketch it here. If the theorem is false, we can choose sequences $\mathcal{P}_i, \delta_i, \mathcal{G}_i$ with the properties (i)–(iii) of that proof, and corresponding simple unital TAF C^*-algebras B_i as well as unital completely positive contractions $\varphi_i, \psi_i : A \longrightarrow B_i$ being δ_i-multiplicative on $\mathcal{G}_i$, satisfying $(\varphi_i)_\sharp(p) = (\psi_i)_\sharp(p)$ for all $p \in \mathcal{P}_i$; yet

$$\inf_{u \in \mathcal{U}(B)} \max_{a \in \mathcal{F}} \|u\varphi_i(a)u^* - \psi_i(a)\| \geq \varepsilon.$$

Define $\dot{\Phi}, \dot{\Psi} : A \longrightarrow \prod B_i / \sum B_i$ and check as in the proof of Theorem 4.1.4 that $\dot{\Phi}$ and $\dot{\Psi}$ are $*$-homomorphisms inducing the same map on $\underline{\mathbf{K}}(A)$. Since (i)–(iii) of PART 1 are met, we conclude from the first part of the proof that there is a unitary $U \in \mathcal{U}(\prod B_i / \sum B_i)$ such that $\left\| U\dot{\Phi}(a)U^* - \dot{\Psi}(a) \right\| < \varepsilon$ for all $a \in \mathcal{F}$. Lifting U to a unitary $(u_i) \in \prod B_i$ and projecting, we get the desired contradiction. ■

Notes 6.1.8 As noted above, Definition 6.1.1 is due to H. Lin. It was motivated by a result of Popa [Pop97] and the classification theory of *AH* algebras ([EG96], [DG97]). Definition 6.1.1 is in fact a version of the original definition, slightly simplified for the class of simple algebras. To correlate this with [Lin98b] one compares to [Lin98b, 3.7] and notes that since Popa's conditions imply the (SP) property (all hereditary subalgebras have a nonzero projection) it suffices to find a p inside a generic corner rather than inside a generic hereditary subalgebra.

6.2 Classification results

We begin this section by presenting a shape type classification result for simple TAF C^*-algebras. Note that in this setting, there is no need to appeal to our existence results, as the KK-classes are represented by completely positive contractions from the outset. We then prove, this time using both existence and uniqueness, that certain TAF C^*-algebras are isomorphic to the AD algebra with the same K-theory. The main result in this section is Theorem 6.2.5. We refer the reader to [CH90] for the basics of asymptotic morphisms.

Theorem 6.2.1 *Let A and B be two unital, separable, nuclear, simple TAF C^*-algebras satisfying the UCT. Suppose that there are unital asymptotic morphisms $\varphi = (\varphi_t) : A \longrightarrow B$ and $\psi = (\psi_t) : B \longrightarrow A$ such that $\varphi_* : \underline{\mathbf{K}}(A) \longrightarrow \underline{\mathbf{K}}(B)$ is bijective and $\varphi_*^{-1} = \psi_*$. Then A is isomorphic to B.*

Proof: We may assume that both A and B are infinite-dimensional. We use repeatedly Theorem 6.1.7 to find an increasing sequence of positive numbers $t_n \in (0, 1)$ and sequences of unitaries $u_{2n-1} \in A$, $u_{2n} \in B$, $n \geq 1$ such that the following diagram is a two-sided approximate intertwining in the sense of Elliott [Ell93].

$$
\begin{tikzcd}
A \arrow[rr, "\mathrm{Ad}\, u_1"] \arrow[dr, "\varphi_{t_1}"'] & & A \arrow[rr, "\mathrm{Ad}\, u_3"] \arrow[dr, "\varphi_{t_3}"'] & & A \arrow[r] \arrow[dr] & \cdots \\
& B \arrow[rr, "\mathrm{Ad}\, u_2"'] \arrow[ur, "\psi_{t_2}"] & & B \arrow[rr, "\mathrm{Ad}\, u_4"'] \arrow[ur, "\psi_{t_4}"] & & B \arrow[r] & \cdots
\end{tikzcd}
$$

■

If r is a positive integer we denote by $\mathbb{I}_r^\sim$ the C^*-algebra of continuous functions $f : [0, 1] \longrightarrow \mathbf{M}_r(\mathbb{C})$ such that $f(0), f(1) \in \mathbb{C}1_r$. We denote by $\mathbb{I}_r$ the subalgebra of functions vanishing at 0. Let $\mathcal{D}$ be the class of algebras B of the form $B = B_1 \oplus \cdots \oplus B_n$ where each B_i is either a circle algebra $\mathbf{M}_k(C(S^1))$ or a dimension-drop algebra $\mathbf{M}_k(\mathbb{I}_r^\sim)$. An AD algebra is a C^*-algebra which is isomorphic to an inductive limit of a sequence of C^*-algebras in $\mathcal{D}$.

Lemma 6.2.2 *Let D be a dimension drop algebra $D = \mathbb{I}_n$ or $D = C_0(\mathbb{R})$. Let E be a simple unital TAF algebra and let f be a nonzero projection in E. Then the map $[D, fEf] \longrightarrow KK(D, E)$ is surjective.*

Proof: Let $n = 1$ if $D = C_0(\mathbb{R})$. The proof uses the following series of facts:

(i) ([DL96b]) $\varinjlim [D, \mathbf{M}_k(E)] = KK(D, E)$.

(ii) ([DL94] when $D = \mathbb{I}_n$) If C is a finite dimensional C*-algebra, and $\eta : D \longrightarrow C$ is a $*$-homomorphism, then the map $d \mapsto \eta(d) \otimes 1_n$ from $D \longrightarrow \mathbf{M}_n(C)$ is null homotopic.

(iii) ([Lor97]) There is a finite subset $\mathcal{F} \subseteq D$ and there is $\varepsilon > 0$ such that if $\alpha, \beta : D \longrightarrow B$ are two $*$-homomorphisms satisfying $\|\alpha(d) - \beta(d)\| < \varepsilon$, then α is homotopic to β.

(iv) ([Lor97]) For any finite subset $\mathcal{F} \subseteq D$ and $\varepsilon > 0$ there is a finite subset $\mathcal{F}_1 \subseteq D$ and there is $\varepsilon_1 > 0$ such that if $\alpha : D \longrightarrow B$ is any completely positive contractive map which is ε_1-multiplicative on $\mathcal{F}_1$, then there exists a $*$-homomorphism $\beta : D \longrightarrow B$ with $\|\alpha(d) - \beta(d)\| < \varepsilon/2$ for all $d \in \mathcal{F}$.

Fix $\mathcal{F}$, ε as in (iii) and let $\mathcal{F}_1$, ε_1 be given by (iv). We may assume that $\mathcal{F} \subseteq \mathcal{F}_1$ and $\varepsilon > \varepsilon_1$. Let $x \in KK(D, E)$. Then by (i) x can be represented by some $*$-homomorphism $\gamma : D \longrightarrow \mathbf{M}_k(E)$. With $A = \mathbf{M}_k(E)$, A is a simple unital TAF C^*-algebra by Lemma 6.1.3. Since E is simple there is $m \geq 1$ such that $[1_E] \leq m[f]$. Consider an approximation of id_A provided by Lemma 6.1.5 applied for the set $\gamma(\mathcal{F}_1)$, $\varepsilon_1/2$, and the integer mnk:

$$\begin{tikzcd} A \arrow[r, equal, "\mathrm{id}_A"] \arrow[dr, "\nu"'] & A \\ & pAp \oplus \mathbf{M}_{mnk}(C) \arrow[u, "\mu"'] \end{tikzcd}$$

Therefore

$$\|\mu\nu(a) - a\| < \varepsilon_1/2 \tag{6.3}$$

for $a \in \gamma(\mathcal{F}_1)$, C is a unital finite dimensional C^*-subalgebra of qAq, $\nu(a) = \omega(a) \oplus (\eta(a) \otimes 1_{mnk})$, $\omega(a) = pap$, ν is ε_1-multiplicative on $\gamma(\mathcal{F}_1)$ and $\eta : A \longrightarrow C$ is a unital completely positive contractive map. We may arrange that $\mu(p) \leq f$. Indeed, from $[p] \leq [q]$ and $[p] + mnk[q] = 1$ we see that $(mnk+1)[p] \leq [1]$. Since $\mu(1) = 1_A$, we obtain $(mnk+1)[\mu(p)] \leq [1_A] = n[1_E] \leq mnk[f]$, hence $[\mu(p)] \leq n[\mu(p)] < [f]$ since $K_0(E)$ is weakly unperforated in the sense of [Bla86] by Lemma 6.1.3. After conjugating μ by a suitable unitary, we obtain $\mu(p) \leq f$.

Let us observe that $\omega\gamma$ and $\eta\gamma$ are ε_1-multiplicative on $\mathcal{F}_1$. By (iv) there are $*$-homomorphisms $\omega' : D \longrightarrow pAp$ and $\eta' : D \longrightarrow C$ such that if we set $\nu' = \omega' \oplus (\eta' \otimes 1_{mn})$, then

$$\|\nu\gamma(d) - \nu'(d)\| < \varepsilon/2 \tag{6.4}$$

for all $d \in \mathcal{F}$. From (6.3) we have $\|\mu\nu\gamma(d) - \gamma(d)\| < \varepsilon_1/2$ for $d \in \mathcal{F}_1$. Combining this with (6.4) we get

$$\|\mu\nu'(d) - \gamma(d)\| < (\varepsilon_1 + \varepsilon)/2 < \varepsilon$$

for all $d \in \mathcal{F}_1 \cap \mathcal{F} = \mathcal{F}$. By (iii) this implies that γ is homotopic to $\mu\nu'$. By (ii) ν' is homotopic to ω' so that γ is homotopic to $\mu\omega'$. We conclude the proof by observing that the image of $\mu\omega'$ is contained in fEf since $\mu\omega'(1) = \mu(p) \leq f$. ■

Lemma 6.2.3 *Let A be a unital C^*-algebra with $(K_0(A), [1_A]) = (\mathbb{Q}, 1)$. Any finite set of projections $\mathcal{P}_0 \subseteq A \otimes \mathcal{K}$ can be complemented to a K_0-triple $(\mathcal{P}_0, \mathcal{G}, \delta)$ with the property that for any unital completely positive contraction $\varphi : A \to A$ which is δ-multiplicative on $\mathcal{G}$, one has $\varphi_\sharp(p) = [p]$ for all $p \in \mathcal{P}_0$.*

Proof: We may write $[p] = \frac{r}{s}[1_A]$, so $s \cdot p \oplus m \cdot 1_A \sim (r+m) \cdot 1_A$ for some $m \geq 0$. When φ is sufficiently multiplicative, we have

$$s\varphi_\sharp(p) + m[1_A] = \varphi_\sharp(s \cdot p \oplus m \cdot 1_A) = \varphi_\sharp((r+m) \cdot 1_A) = (r+m)[1_A].$$

■

The following theorem generalizes a result of Lin [Lin98b] by the fact that it allows non-zero (countable) K_1-groups.

Theorem 6.2.4 *Let A be a unital, separable, nuclear and simple TAF C^*-algebra satisfying the UCT and suppose that $K_0(A) \cong \mathbb{Q}$ as ordered groups. Then A is isomorphic to an AD-algebra.*

Proof: Clearly A is infinite-dimensional, and we may assume that $(K_0(A), [1_A]) \cong (\mathbb{Q}, 1)$ as ordered pointed groups. For any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$ we will find an algebra $B \in \mathcal{D}$, and a $*$-homomorphism $\beta : B \longrightarrow A$ such that $\mathcal{F} \subseteq_\varepsilon \beta(B)$. This will prove the theorem as all elements of $\mathcal{D}$ are semiprojective [Lor97]. The class $\mathcal{D}$ was introduced before Lemma 6.2.2. As noted in Remark 4.1.5, applying Theorem 4.1.4 to fixed $\mathcal{F}$ and ε associated to C^*-algebras A, B with torsion-free divisible K_0-groups results in a K_*-triple $(\mathcal{P}, \mathcal{G}, \delta)$ rather than a general $\underline{\mathbf{K}}$-triple.

Let $(\mathcal{P}_0, \mathcal{G}_0, \delta_0)$ and $(\mathcal{V}, \mathcal{G}_1, \delta_1)$ be a K_0-triple and a K_1-triple, respectively, given by Lemma A.2.5 for the input K_*-triple $(\mathcal{P}, \mathcal{G}, \delta)$. Let $(\mathcal{P}', \mathcal{G}', \delta')$ be a K_*-triple given by Lemma A.2.6 for the input triples $(\mathcal{P}, \mathcal{G}_0, \delta_0)$. and $(\mathcal{V}, \mathcal{G}_1, \delta_1)$. We also may assume that $\mathcal{G}'$ and δ' satisfy the conclusion of Lemma 6.2.3 applied for the unital C^*-algebra A and the set of projections $\mathcal{P}_0$.

By [Ell93] there is an AD algebra D and a group isomorphism $\kappa : K_1(A) \longrightarrow K_1(D)$. By Theorem 5.1.5 there exist completely positive contractive maps $\sigma : A \longrightarrow M_N(D)$ and $\mu : A \longrightarrow M_N(\mathbb{C}1_D)$ which are δ'-multiplicative on $\mathcal{G}'$ and $\sigma_\sharp(p') - \mu_\sharp(p') = \kappa[p']$ for all $p' \in \mathcal{G}'$. Here κ is regarded as an element of $\mathrm{Hom}(K_*(A), K_*(B)) \cong \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$. By the choice of the K_*-triple $(\mathcal{P}', \mathcal{G}', \delta')$ we have that $\sigma_\sharp(u) - \mu_\sharp(u) = \kappa[u]$ for all unitaries $u \in \mathcal{V}$. Note that $\mu_\sharp(u) = 0$ since $K_1(\mathbb{C}) = 0$, so that we have $\sigma_\sharp(u) = \kappa[u]$ for all $u \in \mathcal{V}$.

Recall that $\sigma(1_A) = q$ is a projection, so that if we set $B = qM_N(D)q$, then $\sigma : A \longrightarrow B$ is a unital map. Write B as the inductive limit of an increasing sequence of algebras $B_k \in \mathcal{D}$, and let $j_k : B_k \longrightarrow B$ be the inclusion map. Using the Choi-Effros theorem as in Lemma 4.2 of [DL92] we find a sequence of completely positive contractive maps $\eta_k : B \longrightarrow B_k$ such that $j_k\eta_k$ converges to id_B in the point-norm topology. Choose k large enough so that

$$\sigma_\sharp(u) = (j_k\eta_k\sigma)_\sharp(u) \tag{6.5}$$

for all $u \in \mathcal{V}$.

Consider the group morphism $\kappa^{-1}(j_k)_* : K_1(B_k) \longrightarrow K_1(A)$. There is a a unital $*$-homomorphism $\gamma : B_k \longrightarrow A$ such that $\gamma_* = \kappa^{-1}(j_k)_*$, obtained as follows. Write $B_k = \mathbf{M}_{\ell(1)}(B'_{k1}) \oplus \cdots \oplus \mathbf{M}_{\ell(r)}(B'_{kr})$ where the B'_{ki} are either $C(S^1)$ or $\mathbb{I}^{\sim}_{n(i)}$. Let $q_1 + \cdots + q_r = 1_A$ be a partition of 1_A by nonzero projections. We have $K_0(q_iAq_i) \cong K_0(A) \cong \mathbb{Q}$ as ordered groups. Therefore we find mutually orthogonal projections p_i such that $\ell(i)[p_i] \cong [q_i]$. Since A has cancellation of projections we find a unital inclusion

$$\bigoplus_{i=1}^{r} M_{\ell(i)}(p_iAp_i) \subseteq A.$$

Using Lemma 6.2.2 we find unital $*$-homomorphisms $\gamma_i : B_i \longrightarrow p_iAp_i$ such that $\gamma = \oplus_{i=1}^{r}(\gamma_i \otimes \mathrm{id}_{\ell(i)})$ has the desired property.

Next we want to show that $\gamma \circ (\eta_k\sigma)$ gives an approximate factorization of id_A on $K_*(A)$. More precisely we want that $(\gamma\eta_k\sigma)_\sharp(p) = [p]$ for all $p \in \mathcal{P}$. By virtue of our choice of the K_0 and K_1-triples above, it suffices to show that $(\gamma\eta_k\sigma)_\sharp(p_0) = [p_0]$ for all $p_0 \in \mathcal{P}_0$ and $(\gamma\eta_k\sigma)_\sharp(u) = [u]$ for all $u \in \mathcal{V}$. Using Lemma A.2.4 twice, the definition of γ and (6.5) we have

$$(\gamma\eta_k\sigma)_\sharp(u) = \gamma_*(\eta_k\sigma)_\sharp(u) = \kappa^{-1}(j_k)_*(\eta_k\sigma)_\sharp(u) = \kappa^{-1}(j_k\eta_k\sigma)_\sharp(u) = \kappa^{-1}\sigma_\sharp(u) = [u]$$

for all $u \in \mathcal{V}$. It remains to check that $(\gamma\eta_k\sigma)_\sharp(p_0) = [p_0]$ for all $p_0 \in \mathcal{P}_0$, but this follows from Lemma 6.2.3 by our choice of the $\mathcal{G}'$ and δ'.

Define $\alpha = \eta_k\sigma : A \longrightarrow B_k$. We have seen that $(\gamma\alpha)_\sharp(p) = [p]$ for all $p \in \mathcal{P}$. Therefore by Theorem 4.1.4 there is unitary $u \in A$ such that if $\beta = \mathrm{Ad}\, u \circ \gamma$, then $\|\beta\alpha(a) - a\| < \varepsilon$ for all $a \in \mathcal{F}$, hence $\mathcal{F} \subseteq_\varepsilon \beta(B)$. ∎

Theorem 6.2.5 *Let A, B be unital, separable, nuclear and simple TAF C^*-algebras satisfying the UCT. Suppose that $(K_0(A), [1_A]) \cong (K_0(B), [1_B]) \cong (\mathbb{Q}, 1)$ as ordered groups, and $K_1(A) \cong K_1(B)$. Then A is isomorphic to B.*

Proof: By Theorem 6.2.4 both A and B are isomorphic to simple real rank zero AD algebras. These are classified by their K-theory data as proved by Elliott [Ell93]. ∎

Notes 6.2.6 Lin proved Theorem 6.2.4 for $K_1(A) = 0$ in [Lin98b]. He subsequently, independently from and at about the same time as the present work, generalized his result to allow general K_1-groups in [Lin98a]. This paper also discusses classification of C^*-algebras with other K_0-groups.

6.3 Purely infinite C^*-algebras

The purpose of this section is to demonstrate how our methods can be applied to give the classification result of Kirchberg and Phillips starting from three basic, albeit deep, structural results about purely infinite C^*-algebras. The methods used here are very similar to those used in the finite case, with Cuntz' algebra $\mathcal{O}_2$ playing the role of $\mathbf{M}_n(\mathbb{C})$. The exposition will emphasize this similarity. To make it very clear exactly how much we need to import from the theory of this class of C^*-algebras we collect the required results below.

(I) ([KP98, 2.8]) Any exact, separable and unital C^*-algebra embeds unitally into $\mathcal{O}_2$.

(II) ([Rør93, 3.6]) Let B be an admissible target algebra. If $\varphi, \psi : \mathcal{O}_2 \longrightarrow B$ is a pair of unital $*$-homomorphisms, then for any finite set $\mathcal{F} \subseteq \mathcal{O}_2$ and any $\varepsilon > 0$ there exists $u \in \mathcal{U}(B)$ with $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.

(III) (A variation of [Phi97, 2.4]) Let A be a purely infinite nuclear separable unital C^*-algebra. For any $n \geq 1$, any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there is a projection $p \in A$ with $p^\perp A p^\perp \cong \mathbf{M}_n(\mathcal{O}_2)$ such that there exists an approximate factorization of id_A

$$\begin{tikzcd} A \arrow[rr, equal, "\mathrm{id}_A"] \arrow[dr, "\nu"'] & & A \\ & pAp \oplus \mathbf{M}_n(\mathcal{O}_2) \arrow[ur, hook, "\mu"'] & \end{tikzcd}$$

with $\|\mu\nu(a) - a\| < \varepsilon$ for $a \in \mathcal{F}$, $\nu(a) = pap \oplus (\eta(a) \otimes 1_n)$ is ε-multiplicative on $\mathcal{F}$ with $\eta : A \longrightarrow \mathcal{O}_2$ a unital completely positive contractive map, and μ is a unital $*$-monomorphism whose restriction to pAp is the natural inclusion.

Remark 6.3.1 (I) is needed only for nuclear algebras. We have rephrased (II) and (III) slightly to suit our needs. Rørdam requires that B satisfies

$$\mathrm{cel}(B) < \infty \qquad \mathcal{U}(B)/\mathcal{U}_0(B) \cong K_1(B)$$

and this follows by Definition 2.1.3 as explained in Appendix A.1. Also, Phillips proves (iii) only for $n = 1$, but if we write $\mu : \mathcal{O}_2 \longrightarrow \mathbf{M}_n(\mathcal{O}_2)$ for the map sending x to $\mathrm{Diag}(x, \ldots, x)$ and choose a unital embedding $\iota : \mathbf{M}_n(\mathcal{O}_2) \longrightarrow \mathcal{O}_2$, then (II) shows that a unitary $u \in \mathcal{U}(\mathcal{O}_2)$ exists making the diagram

$$\begin{tikzcd} \mathcal{O}_2 \arrow[rr, "\mathrm{Ad}\, u"] \arrow[dr, "\mu"'] & & \mathcal{O}_2 \\ & \mathbf{M}_n(\mathcal{O}_2) \arrow[ur, "\iota"'] & \end{tikzcd}$$

commute up to ε on $\mathcal{F}$. Thus we may replace $\mathcal{O}_2$ by $\mathbf{M}_n(\mathcal{O}_2)$ without loss of generality (and without using the theorem we are aiming for).

Apart from (I)–(III), all we need to know about a purely infinite C^*-algebra A is that it has real rank zero ([Zha90]), that the canonical maps from $\mathrm{Proj}(A)$ and $\mathcal{U}(A)$ to $K_0(A)$ and $K_1(A)$ are surjective ([Cun81]), and furthermore, that if $p, q \in A$ are nonzero projections, then

$$[p] = [q] \Longrightarrow p \sim q.$$

The latter fact is also from [Cun81]. Note that this shows:

Proposition 6.3.2 *A purely infinite simple unital C^*-algebra is an admissible target (of infinite type).*

Theorem 6.3.3 *Let A, B be unital C^*-algebras with A nuclear separable and B containing a unital copy of $\mathcal{O}_2$, and let $\alpha \in KK(A, B)$. Then for any $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{F}, \delta)$ there exists a completely positive contraction $\sigma : A \longrightarrow B$ which is δ-multiplicative on $\mathcal{F}$ and satisfies $\sigma_\sharp(p) = \alpha_*[p]$ for all $p \in \mathcal{P}$.*

Proof: By (I), A embeds unitally into $\mathcal{O}_2$, so with ι defined as the composite

$$A \hookrightarrow \mathcal{O}_2 \hookrightarrow B$$

we obtain a unital simple embedding. By Lemma 3.1.4, the representation d_ι is absorbing, and it is clearly also quasidiagonal as it commutes with the projections $e_n = n \cdot 1_B$. By Theorem 5.1.4, we find N and $\sigma : A \longrightarrow \mathbf{M}_{2N}(B)$ such that

$$(\sigma)_\sharp(p) - (\gamma_N)_\sharp(p) = \alpha_*[p]$$

where γ_N is a $*$-homomorphism of the form

$$\gamma_N(a) = 0 \oplus N \cdot \iota(a),$$

so that $(\gamma_N)_* = 0$ by the fact that ι factors through $\mathcal{O}_2$. Thus $(\sigma)_\sharp = \alpha_*$ on $\mathcal{P}$, and assuming, as we may (by Lemma 5.1.3), that $\sigma(1)$ is a projection p, we have that $p \in \mathbf{M}_N(B)$ is a subprojection of $N \cdot 1_B$ which is equivalent to a subprojection of $1_B = 1_{\mathcal{O}_2}$ via some unitary u. We may hence replace σ by $u\sigma u^* : A \longrightarrow B$ inducing the same partial map on $\mathcal{P}$. ■

Remark 6.3.4 Kirchberg proved a stronger form of the previous theorem where α is lifted to a $*$-monomorphism.

Theorem 6.3.5 *Let A be a purely infinite separable unital nuclear C^*-algebra satifying the UCT. Then for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exists a $\underline{\mathbf{K}}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property. For any purely infinite simple unital C^*-algebra B, and any two unital completely positive contractions $\varphi, \psi : A \longrightarrow B$ which are δ-multiplicative on $\mathcal{G}$, with $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}$, there exists a unitary $u \in \mathcal{U}(B)$ such that $\|u\varphi(a)u^* - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$.*

Proof: The proof follows closely that of Theorem 6.1.7, and we are only going to indicate the changes needed. Here Part 1 of the proof deals with a pair of $*$-homomorphisms into $B = \prod B_i / \sum B_i$ with each B_i a purely infinite C^*-algebra. Applying first Theorem 4.1.4 and then (III) we get $n, \mathcal{P}, \mathcal{G}, \delta, p, q, \nu$ and μ as in that proof. Then Step 1a applies verbatim as soon as one notes that because they are all nonzero, one has cancellation on all the projections in play by the result of Cuntz mentioned above. In Step 1b one applies (II) to obtain a unitary of fBf which conjugates ψ' to φ' to within $\varepsilon/3$ on $\mathcal{F}$. This suffices to achieve the desired conclusion by the argument given in Theorem 4.1.4. Furthermore, to get that fBf is admissible of infinite type, one uses the same argument to reduce to the case of showing that a corner of a purely infinite C^*-algebra is also admissible of infinite type. This is clear since it is itself purely infinite. Finally, Part 2 of the proof carries through verbatim because of Proposition 6.3.2. ■

Theorem 6.3.6 *([Kir94], [Phi94]) Let A and B be purely infinite separable unital nuclear C^*-algebras satisfying the UCT. Then any isomorphism $\kappa : (K_*(A), [1_A]) \longrightarrow (K_*(B), [1_B])$ is induced by a $*$-isomorphism.*

Proof: We may assume that A and B are in the standard form, that is 1_A and 1_B both represent the zero class in their respective K_0-groups. It follows by [Cun81] that A and B both contain unital copies of $\mathcal{O}_2$. We may thus apply the existence result Theorem 6.3.3 to get maps $\sigma_i : A \longrightarrow B$ and $\tau_i : B \longrightarrow A$ which are increasingly multiplicative on larger and larger sets, and induce κ and κ^{-1}, respectively, on larger and larger subsets of $\underline{\mathbf{K}}\,(A)$ and $\underline{\mathbf{K}}\,(B)$. Arranging this appropriately, we may conclude by the uniqueness result Theorem 6.3.5 that unitaries u_i, v_i exist making

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{Ad}\,u_1"] \arrow[dr, "\sigma_1"'] & & A \arrow[rr, "\mathrm{Ad}\,u_2"] \arrow[dr, "\sigma_2"'] & & A \arrow[r] \arrow[dr] & \cdots & \\
& B \arrow[rr, "\mathrm{Ad}\,v_1"'] \arrow[ur, "\tau_1"] & & B \arrow[rr, "\mathrm{Ad}\,v_2"'] \arrow[ur, "\tau_2"] & & B \arrow[r] & \cdots
\end{tikzcd}$$

an approximate intertwining in the sense of Elliott ([Ell93]). ■

A Appendix on fine K-theoretical points

A.1 K-theory of products

It is well known, although perhaps not as well known as it should be, that the natural map $K_*(\prod B_i) \to \prod K_*(B_i)$ is not an isomorphism in general. In this section we are going to study, for large classes of C^*-algebras B_i, injectivity properties of the natural map

$$\underline{\boldsymbol{\eta}} : \underline{\mathbf{K}}\left(\prod B_i\right) \longrightarrow \prod \underline{\mathbf{K}}\,(B_i)$$

defined by collecting the maps induced by the projections $\pi_i : \prod B_i \longrightarrow B_i$. We then use this information to prove that surprisingly often, $\mathrm{Pext}(-, K_*(\prod B_i / \sum B_i))$ will vanish.

Five quantities

We are going to define five quantities **cco** ("cancellation order"), **pfo** ("perforation order"), $\mathbf{elo}_0$, $\mathbf{elo}_1$ ("element lifting order") and **ipo** ("infinite height perturbation order") in $\mathbb{N} \cup \{\infty\}$ for any unital C^*-algebra B by declaring

$\mathbf{cco}(B) \leq \ell$ if whenever $p, q \in B \otimes \mathcal{K}$, then $[p] = [q] \Longrightarrow p \oplus \ell \cdot 1_B \sim q \oplus \ell \cdot 1_B$

$\mathbf{pfo}(B) \leq \ell$ if for any $x \in K_0(B)$ such that $nx \geq 0$ for some $n > 0$, one has $x + \ell[1_B] \geq 0$.

$\mathbf{elo}_0(B) \leq \ell$ if the canonical map $\mathrm{Proj}(\mathbf{M}_\ell(B)) \longrightarrow K_0(B)$ is surjective

$\mathbf{elo}_1(B) \leq \ell$ if the canonical map $\mathcal{U}_\ell(B) \longrightarrow K_1(B)$ is surjective

$\mathbf{ipo}(B) \leq \ell$ if for any x in $K_0(B)$ and any $n \neq 0$, there is y in $K_0(B)$ such that $-\ell[1_B] \leq y \leq \ell[1_B]$ and $x - y \in nK_0(B)$.

and declaring the value to be ∞ when no such ℓ exists. When $\{B_i\}_{i\in I}$ is a family of C^*-algebras, we define $\mathbf{cco}(\{B_i\}) = \sup_i \mathbf{cco}(B_i)$, and so forth. We also write $\mathbf{rr}(\{B_i\}) = 0$ when each B_i has real rank zero.

The following results – some of which are known, cf. [EL] – demonstrate the relevance of these quantities to the map $\underline{\boldsymbol{\eta}}$. We denote the unit of B_i by 1_i and the units of $\prod B_i$ and $\prod B_i/\sum B_i$ by 1_Π and $1_{\Pi/\Sigma}$, respectively.

Lemma A.1.1 *Let B_i be a sequence of unital C^*-algebras for which $\mathbf{cco}(\{B_i\}) < \infty$. Then η^0 is injective, and the image of η^0 equals*

$$\left\{(x_i) \in \prod K_0(B_i) \middle| \exists M \in \mathbb{N} \forall i \in \mathbb{N} : -M[1_i] \leq x_i \leq M[1_i]\right\}.$$

Furthermore, if $\eta^0(x) = (x_i)$ with each $x_i \geq 0$, then $x \geq 0$. If in addition $\mathbf{pfo}(\{B_i\}) < \infty$, then $\operatorname{Im}\eta^0$ is a pure subgroup of $\prod K_0(B_i)$. If $\mathbf{elo}_0(\{B_i\}) < \infty$, then η^0 is surjective.

Proof: Assume that $\mathbf{cco}(\{B_i\}) \leq \ell$. To prove injectivity, let $x = [(p_i)] - [(q_i)] \in K_0(\prod B_i)$ be given by $p_i, q_i \in \mathbf{M}_N(B_i)$ and assume that $\eta^0(x) = 0$. We have that $p_i \oplus \ell\cdot 1_i \sim q_i \oplus \ell \cdot 1_i$, wherefrom $(p_i) \oplus \ell\cdot 1_\Pi \sim (q_i) \oplus \ell\cdot 1_\Pi$, proving $x = 0$. To prove that the image is contained in the set of bounded sequences, write $x \in K_0(\prod B_i)$ as $x = [(p_i)] - [(q_i)]$ with $p_i, q_i \in \mathbf{M}_N(B_i)$. By definition of positivity, $-N[1_\Pi] \leq x \leq N[1_\Pi]$ and we get the result by applying π_i. For the other inclusion, assume that $x_i \in K_0(B_i)$ is given with $-M[1_i] \leq x_i \leq M[1_i]$. We can write $x_i + M[1_i] = [r_i]$ for some projection r_i in $B_i \otimes \mathcal{K}$, and since $[r_i] \leq 2M[1_i]$ we get that $2M[1_i] = [r_i \oplus s_i]$ for some projection s_i in $B_i \otimes \mathcal{K}$. By $\mathbf{cco}(\{B_i\}) \leq \ell$ we see that $(2M + \ell)\cdot 1_i \sim r_i \oplus s_i \oplus \ell\cdot 1_i$, hence there is $q_i \in \mathbf{M}_{2M+\ell}(B)$ with $r_i \sim q_i$. Consequently, $x_i = [r_i] - M[1_i]$ can be represented as a difference of projections $[q_i] - M[1_i]$ where $q_i \in \mathbf{M}_{2M+\ell}(B_i)$. Defining $q = (q_i) \in \mathbf{M}_{2M+\ell}(\prod B_i)$, $[q] - M[1_\Pi]$ is a preimage of (x_i).

If $\eta^0(x) = (x_i)$ and every x_i is positive, we have $0 \leq x_i \leq M[1_i]$ for some fixed M. Hence $x_i = [p_i]$ for some p_i which we may assume lies in $M_{M+\ell}(B_i)$ as above. Consequently $p = (p_i)$ defines an element of $K_0(\prod B_i)$, and $x = [p]$ by injectivity of η^0. To establish purity when $\mathbf{pfo}(\{B_i\}) \leq \ell'$, assume that $x = ny$ in $\prod K_0(B_i)$, where $x \in \operatorname{Im}\eta^0$ so that for some M, $Mn[1_i] \pm x_i \geq 0$. We conclude that $(M + \ell')[1_i] \pm y_i \geq 0$, whence $y \in \operatorname{Im}\eta^0$. Proving surjectivity of η^0 when $\mathbf{elo}_0(\{B_i\})$ is finite is straightforward. ■

Lemma A.1.2 *Let B_i is a sequence of C^*-algebras. If $\mathbf{rr}(\{B_i\}) = 0$, then η^1 is injective. If $\mathbf{elo}_1(\{B_i\}) < \infty$, then η^1 is surjective.*

Proof: To prove injectivity, we assume that $\eta^1(x) = 0$ with $x = [(u_i)]$ and $u_i \in \mathbf{M}_N(B_i)$. By [Lin96], u_i is homotopic to $n\cdot 1_i$ within $\mathcal{U}_n(B_i)$, and since $\mathbf{M}_n(B_i)$ has finite exponential length because $\mathbf{rr}(B_i) = 0$ (see [Lin93]), we can combine these paths to one from (u_n) to the unit of $\mathcal{U}_n(\prod B_i)$. Proving surjectivity of η^1 when $\mathbf{elo}_1(\{B_i\})$ is finite is straightforward. ■

Lemma A.1.3 *Let B_i be a sequence of unital C^*-algebras and abbreviate $\Pi = \prod B_i$, $\Pi/\Sigma = \prod B_i/\sum B_i$. Then*

(i) $\mathbf{cco}(\Pi), \mathbf{cco}(\Pi/\Sigma) \leq \mathbf{cco}(\{B_i\})$.
(ii) $\mathbf{pfo}(\Pi), \mathbf{pfo}(\Pi/\Sigma) \leq \mathbf{pfo}(\{B_i\})$ *if* $\mathbf{cco}(\{B_i\}) < \infty$.
(iii) $\mathbf{elo}_0(\Pi), \mathbf{elo}_0(\Pi/\Sigma) \leq \mathbf{elo}_0(\{B_i\})$ *if* $\mathbf{cco}(\{B_i\}) < \infty$.
(iv) $\mathbf{elo}_1(\Pi), \mathbf{elo}_1(\Pi/\Sigma) \leq \mathbf{elo}_1(\{B_i\})$ *if* $\mathbf{rr}(\{B_i\}) = 0$.
(v) $\mathbf{ipo}(\Pi), \mathbf{ipo}(\Pi/\Sigma) \leq \mathbf{ipo}(\{B_i\})$ *if* $\mathbf{cco}(\{B_i\}), \mathbf{pfo}(\{B_i\}) < \infty$.

Proof: These claims all follow in a straightforward fashion from the properties of η^* established in Lemma A.1.1 and A.1.2. We only prove (v), which is the most involved result. Let $x \in K_0(\prod B_i)$ and n be given and assume that $\mathbf{ipo}(\{B_i\}) \leq \ell$. With $\eta^0(x) = (x_i)$, we can find $y_i \in K_0(B_i)$ with $-\ell[1_i] \leq y \leq \ell[1_i]$ and $x_i - y_i \in nK_0(B_i)$. We know from Lemma A.1.1 that $(y_i) = \eta^0(y)$ for some $y \in K_0(\prod B_i)$ with $-\ell[1_\Pi] \leq y \leq \ell[1_\Pi]$. Since $\eta^0(x-y) \in n \prod K_0(B_i)$ by construction, we further conclude by purity that $\eta^0(x-y) \in n \operatorname{Im} \eta^0$ and, since η^0 is injective, that $x - y \in nK_0(\prod B_i)$.

To prove the result for Π/Σ, we note that

$$0 \longrightarrow K_*(\Sigma) \longrightarrow K_*(\Pi) \longrightarrow K_*(\Pi/\Sigma) \longrightarrow 0$$

is exact and apply the argument above to a $x \in K_0(\Pi)$ lifting the given element in $K_0(\Pi/\Sigma)$. ■

Algebraically compact K-groups

An abelian group G is *algebraically compact* when $\mathrm{Pext}(-, G)$ vanishes. This class of groups is well studied (cf. [Fu70, VII]), and we are going to use the characterization of it as those groups for which

(i) The subgroup $\bigcap_{n \in \mathbb{N}} nG$ is divisible.
(ii) G is complete

hold ([Hul62a]). Here completeness refers to the $\mathbb{Z}$-adic topology (cf. [Fu70, 7]), and completeness does *not* (as it does in [Fu70]!) imply any separation properties.

Note that the quantities **pfo** and **ipo** make sense for general ordered abelian groups with order unit. We extend the notions to such groups and families of them in the obvious way. When $(G_i, 1_i)$ is a family of groups and order units, we set

$$\prod_b G_i = \{(g_i) | \, \exists M \in \mathbb{N} \, \forall i \in I : -M1_i \leq g_i \leq M1_i\}.$$

Lemma A.1.4 *Whenever G_i is a sequence of abelian groups, then $\prod G_i / \sum G_i$ is algebraically compact. If, furthermore, all G_i are ordered with order units, then $\prod_b G_i / \sum G_i$ is algebraically compact provided that both* $\mathbf{pfo}(\{G_i\})$ *and* $\mathbf{ipo}(\{G_i\})$ *are finite.*

Proof: When $X = \prod G_i / \sum G_i$, then X is algebraically compact by [Hul62b]. When the G_i are also ordered, let $X_b = \prod_b G_i / \sum G_i$ and consider X_b as a subgroup of X. We are going to prove that (i) and (ii) above hold for X_b from the fact that they hold for X.

Since $\mathbf{pfo}(\{G_i\}) < \infty$, we immediately get that if $x \in X_b$ and $x = my$ in X, then $y \in X_b$ if $m \neq 0$, by an argument very similar to the one at the end of proof of Lemma A.1.1. Note that this is stronger than the purity of X_b as a subgroup of X. Fix $m \neq 0$ and $x \in \bigcap_{n\in\mathbb{N}} nX_b$ and write $x = my$ for $y \in \bigcap_{n\in\mathbb{N}} nX$. Applying this observation twice, we get

$$y \in X_b \cap \bigcap_{n\in\mathbb{N}} nX = \bigcap_{n\in\mathbb{N}} (nX \cap X_b) = \bigcap_{n\in\mathbb{N}} nX_b.$$

It remains to show that X_b is complete. To do so, we first note that for any given $x \in X$ there is $y \in X_b$ such that $x - y \in \bigcap_{n\in\mathbb{N}} nX$. For when $\mathbf{ipo}(\{G_i\}) \leq \ell$ and $x = (x_i) + \sum G_i$ is given, we may find y_i such that $-\ell 1_i \leq y_i \leq \ell 1_i$ and $x_i - y_i \in i!G_i$. Clearly $y = (y_i) + \sum G_i$ has the desired property. Now let (x_n) be Cauchy in X_b, and recall that it converges to some $x \in X$. With y chosen as above, (x_n) also converges to $y \in X_b$ in the $\mathbb{Z}$-adic topology of X_b. This is because the $\mathbb{Z}$-adic topology of X_b coincides with the topology of X_b induced by the $\mathbb{Z}$-adic topology of X, since X_b is a pure subgroup of X. ■

Corollary A.1.5 *Let B_i be a sequence of unital C^*-algebras.*

(i) $K_1(\prod B_i / \sum B_i)$ is algebraically compact if $\mathbf{rr}(\{B_i\}) = 0$ and $\mathbf{elo}_1(\{B_i\}) < \infty$.
(ii) $K_0(\prod B_i / \sum B_i)$ is algebraically compact if either $\mathbf{cco}(\{B_i\}), \mathbf{elo}_0(\{B_i\}) < \infty$ or $\mathbf{cco}(\{B_i\}), \mathbf{pfo}(\{B_i\}), \mathbf{ipo}(\{B_i\}) < \infty$.

Proof: Part (i) follows from Lemmas A.1.2 and A.1.4. Part (ii) follows from Lemmas A.1.1 and A.1.4. ■

Admissible targets

We are now ready to collect our results in the special case of admissible target algebras.

Theorem A.1.6 *Let B_i be a sequence of admissible target algebras of the same type and let A be any C^*-algebra. Then*

(i) $\underline{\boldsymbol{\eta}} : \underline{\mathbf{K}}(\prod B_i) \longrightarrow \prod \underline{\mathbf{K}}(B_i)$ is injective.
(ii) The natural map $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(\prod B_i)) \longrightarrow \prod \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B_i))$ is injective.
(iii) The natural map $KK(A, \prod B_i / \sum B_i) \longrightarrow \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(\prod B_i / \sum B_i))$ is an isomorphism if A satisfies the UCT.
(iv) $\prod B_i$ and $\prod B_i / \sum B_i$ are admissible targets

Proof: For (i), decompose $\underline{\boldsymbol{\eta}}$ into maps η^* and η_n^*. We get from Lemma A.1.1 and A.1.2 that η^* are injections whose images are pure subgroups. Injectivity of η_n^* then follows by a diagram chase on

$$\begin{tikzcd}
K_*(\prod B_i) \arrow[r, "\times n"] \arrow[d, "\eta^*"'] & K_*(\prod B_i) \arrow[r, "\rho_n^*"] \arrow[d, "\eta^*"'] & K_*(\prod B_i; \mathbb{Z}/n) \arrow[r, "\beta_n^*"] \arrow[d, "\eta_n^*"'] & K_{*+1}(\prod B_i) \arrow[d, "\eta^{*+1}"'] \\
\prod K_*(B_i) \arrow[r, "\times n"'] & \prod K_*(B_i) \arrow[r, "\prod \rho_n^*"'] & \prod K_*(B_i; \mathbb{Z}/n) \arrow[r, "\prod \beta_n^*"'] & \prod K_{*+1}(B_i).
\end{tikzcd}$$

Claim (ii) is a direct consequence of (i), and (iii) follows by combining the UMCT 2.1 with Corollary A.1.5. Finally, (iv) follows by Lemma A.1.3. ■

A.2 Partial maps on $\underline{\mathbf{K}}(-)$

In this appendix we concern ourselves with associating K-theoretical data to completely positive contractions. Starting from such maps, say from A to B, we shall be able to induce maps sending a finite set of projections representing a finite part of the K-theory of A to elements of the K-theory of B.

Although there are advantages of doing this even for subsets representing elements of $K_0(A)$, the real strength of this approach only surfaces when we work with all of $\underline{\mathbf{K}}(A)$ and $\underline{\mathbf{K}}(B)$. Our *partial maps* do not descend to well-defined maps on subsets of $\underline{\mathbf{K}}(A)$, let alone to all of $\underline{\mathbf{K}}(A)$, but this fact does not cause any problems except notational and technical inconveniences.

As noted in [DG97] we can realize any element of $\underline{\mathbf{K}}(-)$ as a difference of projections from

$$\underline{\mathrm{Proj}}(A) = \bigcup_{m\geq 1} \mathrm{Proj}(A \otimes C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K})$$

where the W_m are the Moore spaces of order m. This picture of $\underline{\mathbf{K}}(A)$ encompasses the standard pictures of $K_0(A)$ and $K_*(A)$ using projections of $A \otimes \mathcal{K}$ and $A \otimes C(\mathbb{T}) \otimes \mathcal{K}$, respectively, but not the standard picture of $K_1(A)$ using unitaries of $(A \otimes \mathcal{K})^\sim$. We need to pay special attention to this. Checking the facts stated as lemmas below is tedious but straight-forward. We leave it to the reader with due apologies.

Definition A.2.1 Let A be a C^*-algebra. A $\underline{\mathbf{K}}$*-triple* $(\mathcal{P}, \mathcal{G}, \delta)$ consists of finite subsets $\mathcal{P} \subseteq \underline{\mathrm{Proj}}(A)$ and $\mathcal{G} \subseteq A$ and a $\delta > 0$ chosen such that whenever φ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$, then

$$\tfrac{1}{2} \notin \mathrm{sp}((\varphi \otimes \mathrm{id})(p))$$

for each $p \in \mathcal{P}$, where id is the identity of $C(\mathbb{T}) \otimes C(W_m) \otimes \mathcal{K}$ for suitable m. A K_1*-triple* $(\mathcal{V}, \mathcal{G}, \delta)$ consists of finite subsets $\mathcal{V} \subseteq \mathcal{U}((A \otimes \mathcal{K})^\sim)$ and $\mathcal{G} \subseteq A$ and a $\delta > 0$ chosen such that whenever φ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$, then

$$0 \notin \mathrm{sp}((\varphi \otimes \mathrm{id})(v))$$

for each $v \in \mathcal{V}$.

We define K_0- and K_*-triples analogously to the $\underline{\mathbf{K}}$-triple case, by using projections in $A \otimes \mathcal{K}$ and $A \otimes C(\mathbb{T}) \otimes \mathcal{K}$, respectively.

The following lemma shows that any finite subset of projections or unitaries can be *complemented* to a triple of the appropriate kind.

Lemma A.2.2 *Let A and C be C^*-algebras, and fix finite sets $\mathcal{P}, \mathcal{V} \subseteq (A \otimes C)^\sim$ consisting of projections and unitaries, respectively. Then there exists $\delta > 0$ and a finite set $\mathcal{G} \subseteq A$ such that whenever $\varphi : A \longrightarrow B$ is a unital completely positive map which is δ-multiplicative on $\mathcal{G}$, then*

$$\tfrac{1}{2} \notin \mathrm{sp}((\varphi \otimes \mathrm{id}_C)(p)) \qquad 0 \notin \mathrm{sp}((\varphi \otimes \mathrm{id}_C)(v))$$

for every $p \in \mathcal{P}$, $v \in \mathcal{V}$.

Let $\chi_0 : [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \to [0, 1]$ be 0 on $[0, \frac{1}{2})$ and 1 on $(\frac{1}{2}, 1]$, and let $\chi_1 : (0, 1] \to \mathbb{C}$ be given by $\chi_1(x) = x^{-1/2}$.

Definition A.2.3 Let $(\mathcal{P}, \mathcal{G}, \delta)$ be a $\underline{\mathbf{K}}$-triple, and assume that $\varphi : A \longrightarrow B$ is a completely positive contraction which is δ-multiplicative on $\mathcal{G}$. We define $\varphi_\sharp : \mathcal{P} \longrightarrow \underline{\mathbf{K}}(B)$ by

$$\varphi_\sharp(p) = [\chi_0(\varphi \otimes \mathrm{id})(p)].$$

When $(\mathcal{V}, \mathcal{G}, \delta)$ is a $\underline{\mathbf{K}}$-triple, we define $\varphi_\sharp : \mathcal{V} \to K_1(B)$ by

$$\varphi_\sharp(v) = [V\chi_1(VV^*))]$$

where $V = (\varphi \otimes \mathrm{id})(v)$.

Maps into $K_0(B)$ and $K_*(B)$ are defined from K_0-triples and K_*-triples similarly to the $\underline{\mathbf{K}}$-triple case.

Lemma A.2.4 *Let A be a unital C^*-algebra and $(\mathcal{P}, \mathcal{G}, \delta)$ a $\underline{\mathbf{K}}$-triple. Let $\varphi : A \longrightarrow B$ be a completely positive map which is δ-multiplicative on $\mathcal{G}$ and let $j : B \longrightarrow C$ be a unital $*$-homomorphism. Then $(j\varphi)_\sharp(p) = j_*\varphi_\sharp(p)$ for all $p \in \mathcal{P}$.*

To establish the next two results, one may use that the canonical map from $K_0(A)$ to $K_1(SA)$ is defined using *scalar* rotation matrices.

Lemma A.2.5 *Whenever a K_*-triple $(\mathcal{P}_*, \mathcal{G}, \delta)$ is given, there exist a K_0-triple $(\mathcal{P}_0, \mathcal{G}_0, \delta_0)$ and a K_1-triple $(\mathcal{V}, \mathcal{G}_1, \delta_1)$ with $\delta_i < \delta$ and $\mathcal{G}_i \supseteq \mathcal{G}$ such that if for two unital completely positive contractions φ, ψ which are δ_i-multiplicative on $\mathcal{G}_i$,*

$$\varphi_\sharp(p) = \psi_\sharp(p) \qquad \varphi_\sharp(v) = \psi_\sharp(v)$$

for all $p \in \mathcal{P}_0$ and all $v \in \mathcal{V}$, then $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}_*$.

Lemma A.2.6 *Whenever a K_0-triple $(\mathcal{P}_0, \mathcal{G}_0, \delta_0)$ and a K_1-triple $(\mathcal{V}, \mathcal{G}_1, \delta_1)$ is given, there exists a K_*-triple $(\mathcal{P}_*, \mathcal{G}, \delta)$ with $\delta < \delta_i$ and $\mathcal{G} \supseteq \mathcal{G}_i$ such that if for two unital completely positive contractions φ, ψ which are δ-multiplicative on $\mathcal{G}$, $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}_*$, then*

$$\varphi_\sharp(p) = \psi_\sharp(p) \qquad \varphi_\sharp(v) = \psi_\sharp(v)$$

for all $p \in \mathcal{P}_0$ and all $v \in \mathcal{V}$.

Lemma A.2.7 *Let $(\mathcal{P}, \mathcal{G}, \delta)$ be a $\underline{\mathbf{K}}$-triple on A. There exists $\varepsilon > 0$ and a finite set $\mathcal{F} \subseteq A$ such that if φ, ψ are completely positive contractions which are δ-multiplicative on $\mathcal{G}$ and satisfy $\|\varphi(a) - \psi(a)\| < \varepsilon$ for all $a \in \mathcal{F}$, then $\varphi_\sharp(p) = \psi_\sharp(p)$ for all $p \in \mathcal{P}$.*

References

[BK97] B. Blackadar and E. Kirchberg, *Generalized inductive limits of finite-dimensional C^*-algebras*, Math. Ann. **307** (1997), no. 3, 343–380.

[Bla86] B. Blackadar, *K-theory for operator algebras*, Math. Sci. Research Inst. Publ., vol. 5, Springer-Verlag, New York, 1986.

[Bra72] O. Bratteli, *Inductive limits of finite-dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[BDF77] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. **105** (1977), no. 2, 265–324.

[BP91] L.G. Brown and G.K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.

[Bro77] L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), no. 2, 335–348.

[CE76] M.D. Choi and E.G. Effros, *The completely positive lifting problem for C^*-algebras*, Ann. of Math. (2) **104** (1976), no. 3, 585–609.

[CH90] A. Connes and N. Higson, *Déformations, morphismes asymptotiques et K-théorie bivariante*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 2, 101–106.

[Cun81] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181–197.

[Dar97] M. Dădărlat, *Nonnuclear subalgebras of AF algebras*, preprint 1997.

[DG97] M. Dădărlat and G. Gong, *A classification result for approximately homogeneous C^*-algebras of real rank zero*, Geom. Funct. Anal. **7** (1997), no. 4, 646–711.

[DL92] M. Dădărlat and T.A. Loring, *The K-theory of Abelian subalgebras of AF algebras*, J. reine angew. Math. **432** (1992), 39–55.

[DL94] M. Dădărlat and T.A. Loring, *Extensions of certain real rank zero C^*-algebras*, Ann. Inst. Fourier **44** (1994), 907–925.

[DL96a] M. Dădărlat and T.A. Loring, *Classifying C^*-algebras via ordered, mod-p K-theory*, Math. Ann. **305** (1996), no. 4, 601–616.

[DL96b] M. Dădărlat and T.A. Loring, *A universal multicoefficient theorem for the Kasparov groups*, Duke Math. J. **84** (1996), no. 2, 355–377.

[EG96] G.A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II*, Ann. of Math. **144** (1996), 497–610.

[EL] S. Eilers and T.A. Loring, *Computing contingencies for stable relations*, *Internat. J. Math.*, to appear.

[Ell76] G.A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[Ell93] G.A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. reine angew. Math. **443** (1993), 179–219.

[Fu70] L. Fuchs, *Infinite abelian groups I*, Academic Press, New York, San Francisco, London, 1970.

[Hig87a] N. Higson, *A characterization of KK-theory*, Pacific J. Math. **126** (1987), no. 2, 253–276.

[Hig87b] N. Higson, *On a technical theorem of Kasparov* , J. Funct. Anal. **73** (1987), no. 1, 107–112

[Hig90] N. Higson, *Categories of fractions and excision in KK-theory*, J. Pure Appl. Algebra **65** (1990), no. 2, 119–138.

[Hig95] N. Higson, *C^*-algebra extension theory and duality*, J. Funct. Anal. **129** (1995), no. 2, 349–363.

[HLT96] T.G. Houghton-Larsen and K. Thomsen, *Universal (co)homology theories*, preprint, 1996.

[Hul62a] A. Hulanicki, *On algebraically compact groups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 71–75.

[Hul62b] A. Hulanicki, *The structure of the factor group of the unrestricted sum by the restricted sum of Abelian groups*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 77–80.

[JT91] K.K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, Boston, 1991.

[Kas80a] G.G. Kasparov, *Hilbert C^*-modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), no. 1, 133–150.

[Kas80b] G.G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636, 719.

[Kir94] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, preprint, third draft, 1994.

[KP98] E. Kirchberg and N.C. Phillips, *Embedding of exact C^*-algebras and continuous fields in the Cuntz algebra $\mathcal{O}_2$*, preprint, 1998.

[Lin93] H. Lin, *Exponential rank of C^*-algebras with real rank zero and the Brown-Pedersen conjectures*, J. Funct. Anal. **114** (1993), no. 1, 1–11.

[Lin96] H. Lin, *Approximation by normal elements with finite spectra in C^*-algebras of real rank zero*, Pacific J. Math. **173** (1996), no. 2, 443–489.

[Lin97] H. Lin, *Stably approximately unitary equivalence of homomorphisms*, preprint, 1997.

[Lin98a] H. Lin, *Classification of simple tracially AF C^*-algebras*, preprint, 1998.

[Lin98b] H. Lin, *Tracially AF C^*-algebras*, preprint, 1998.

[Lor97] T.A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs, vol. 8, American Mathematical Society, Providence, RI, 1997.

[LP95] H.X. Lin and N.C. Phillips, *Almost multiplicative morphisms and the Cuntz algebra $\mathcal{O}_2$*, Internat. J. Math. **6** (1995), no. 4, 625–643.

[Pas81] W.L. Paschke, *K-theory for commutants in the Calkin algebra*, Pacific J. Math. **95** (1981), no. 2, 427–434.

[Ped79] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[Phi94] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Preprint, 1994.

[Phi97] N.C. Phillips, *Approximate unitary equivalence of homomorphisms from odd Cuntz algebras*, Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 243–255.

[Pop97] S. Popa, *On local finite-dimensional approximation of C^*-algebras*, Pacific J. Math. **181** (1997), no. 1, 141–158.

[Rie83] M.A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. **46** (1983), no. 3, 301–333.

[Rie87] M.A. Rieffel, *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory **17** (1987), no. 2, 237–254.

[Rør93] M. Rørdam, *Classification of inductive limits of Cuntz algebras*, J. reine angew. Math. **440** (1993), 175–200.

[Rør95] M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal. **131** (1995), no. 2, 415–458

[RS87] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), 431–474.

[Sal92] N. Salinas, *Relative quasidiagonality and KK-theory*, Houston J. Math. **18** (1992), no. 1, 97–116

[Sch84] C. Schochet, *Topological methods for C^*-algebras IV: Mod p homology*, Pacific J. Math. **114** (1984), 447–468.

[Sch84] C. Schochet, *The fine structure of the Kasparov groups II: relative quasidiagonality*, Preprint.

[Ska88] G. Skandalis, *Une notion de nucléarité en K-théorie (d'après J. Cuntz)*, K-Theory **1** (1988), no. 6, 549–573.

[Val83] A. Valette, *A remark on the Kasparov groups* $\mathrm{Ext}(A, B)$, Pacific J. Math. **109** (1983), no. 1, 247–255.

[Voi76] D. Voiculescu, *A non-commutative Weyl–von Neumann theorem*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.

[Voi93] D. Voiculescu, *Around quasidiagonal operators*, Integral Equations Operator Theory **17** (1993), no. 1, 137–149.

[Zha90] S. Zhang, *A property of purely infinite simple C^*-algebras*, Proc. Amer. Math. Soc. **109** (1990), no. 3, 717–720.

Marius Dădărlat
Department of Mathematics
Purdue University
West Lafayette
IN 47907
U.S.A.
mdd@math.purdue.edu

Søren Eilers
Matematisk Afdeling
Københavns Universitet
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
eilers@math.ku.dk